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Symmetries of $\mathcal{N} = 4$ SYM and the Regge limit of gauge theories

Memoria de Tesis doctoral

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presentada ante el Departamento de Física Teórica
de la Universidad Autónoma de Madrid
para la obtención del título de Doctor.

Proyecto dirigido por

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Madrid, Septiembre 2010

Resumen y Conclusiones

Una de las herramientas más importantes para la física teórica moderna es el estudio de la simetría. Esta tesis trata dos contextos muy distintos en los que se han logrado grandes avances debido a un análisis de simetrías. El primero es el del espectro de dimensiones anómalas en $\mathcal{N} = 4$ Super Yang Mills, que en los últimos años parece haber sido completamente determinado, debido a un modelo integrable subyacente. El segundo es el del límite Regge de teorías gauge, donde surge también una simetría integrable de dimensión infinita, a pesar de que hayamos partido de una teoría, como QCD, con una cantidad finita de simetría. En la tesis se presenta el descubrimiento de dos simetrías nuevas, que potencialmente podrían aumentar nuestra comprensión sobre estos dos contextos.

Para empezar se plantea la posibilidad de acceder a la energía del magnon sencillo, componente esencial del modelo integrable que determina el espectro de dimensiones anómalas en $\mathcal{N} = 4$, haciendo una continuación analítica del espectro en el sector $SL(2)$. Esto permite sacar conclusiones sobre la energía del magnon fuera del rango de validez de su relación de dispersión asintótica. A continuación, este método se aplica para motivar una simetría nueva en la teoría β-deformada de $\mathcal{N} = 4$, relacionando los espectros para distintos valores de la deformación en el sector $SU(2)$. Como aplicación de esta nueva simetría, se muestra como impone ligaduras para la estructura del operador de dilataciones, incluyendo efectos de wrapping, en la teoría $\mathcal{N} = 4$ original.

Después se constata la existencia de una simetría $SL(2, C)$ nueva en la ecuación BFKL, muy parecida a la simetría dual conforme de las amplitudes de scattering en $\mathcal{N} = 4$, y con potencial para explicar la integrabilidad del límite Regge de QCD. La simetría nueva parece romperse debido a efectos infrarrojos, pero se muestra que, por lo menos al orden más bajo en la ruptura de la simetría, ésta se puede recuperar deformando su representación.

Abstract

One of the most important tools in modern theoretical physics is that of symmetry. This thesis is concerned with two areas that, despite being very different, have seen an impressive amount of development due to an analysis of their symmetries. The first is the spectrum of anomalous dimensions in $\mathcal{N} = 4$ Super Yang Mills, which very recently has possibly been determined completely, due to the formulation of an underlying integrable model. The second area is the Regge limit of gauge theories, where an infinite-dimensional integrable symmetry also emerges, even though one starts with a theory, such as QCD, with a finite amount of symmetry. In the thesis two new symmetries are presented, which have the potential to increase our understanding of both of these areas.

First, we make the observation that one can access the energy of the single magnon, an essential building block in the integrable model for the spectrum of anomalous dimensions in $\mathcal{N} = 4$, by performing an analytical continuation of the spectrum in the $SL(2)$-sector. This permits us to draw conclusions about the magnon energy outside of the regime of validity of its asymptotic dispersion relation. Next, this method is used to motivate a new symmetry in β-deformed $\mathcal{N} = 4$, relating the spectrum of the $SU(2)$ sector at different values of the deformation parameter β. As an application of this symmetry, we show how it imposes constraints on the structure of the dilatation operator, including wrapping effects, in the original $\mathcal{N} = 4$ theory.

We then note the existence of a new dual $SL(2, C)$ symmetry of the BFKL equation, closely analogous to the dual conformal symmetry of scattering amplitudes in $\mathcal{N} = 4$, and having the potential to explain the integrability found in the Regge limit. This new symmetry would seem to be broken by IR effects, but it is shown that, at least to lowest order in the symmetry breaking, it can be recovered by a deformation of its representation.

Acknowledgments

First and foremost, I would like to thank my supervisor César Gómez for all the support and understanding he has given me over these past four years, and for introducing me to a fascinating field of research, while at the same time being refreshingly down-to-earth. I am also indebted to Agustín Sabio Vera for answering numerous questions, as well as helping me by proofreading and providing comments on one of my articles, and to Rafael Hernández for the same reasons, as well as for helping me with the bureaucracy surrounding this thesis.

I would also like to thank the members of the IFT and theoretical physics department at UAM, for creating an interesting, stimulating and above all welcoming environment. Special thanks go to my fellow phd students, Luis Aparicio, Ana Rodriguez, Norberto Gutiérrez, José Roberto Vidal Madrid, Javier Abajo, Javier Rubio, Pablo Soler, João Aparício, Josu Cantero, Pilar Coloma, Francisco Peña, as well as the ex-phd students Sergio Montañez, Carlos Fuertes and Daniel Figueroa for making the time spent at the IFT a particularly pleasant one.

And of course I am grateful to my fantastic wife Marta, for all her loving support and patience. I am also grateful to my daughter Angela who, despite not having contributed in a positive sense to this work, has allowed me to keep my sanity.

This work is supported by a Spanish FPU grant.

Contents

Chapter 1

Introduction

Symmetry has become one of the most important tools of modern physics. When constructing a quantum field theory, the most successful framework available for describing the interactions of elementary particles, the modern viewpoint is to start by assuming a certain amount of symmetry and then adding all possible interactions compatible with that symmetry. In other cases, symmetry is not an input into the construction of a theory but is rather discovered afterwards. Such unexpected symmetries are very important since they constrain the possible solutions of the theory, and in some extreme cases, the so-called integrable theories, allow for an exact solution of the theory. What is meant by an exact solution is a formula, or a set of equations, which provide the values for all relevant physical observables, allowing us to obtain important insights into otherwise inaccessible areas of physics. In this thesis we will be working with a quantum field theory which is believed to be integrable, the conformally invariant four-dimensional gauge theory $\mathcal{N} = 4$ Super Yang Mills. This theory, and some of its deformations, are believed to be the first known examples of four-dimensional integrable quantum field theories. Even though the theory lacks some realism it is hoped that it can provide important insights into the behavior of four-dimensional quantum field theories in general.

Another cornerstone of contemporary modern theoretical physics is that of string theory. Originally, the theory was conceived as a dual description of hadronic physics, where the so-called Regge trajectories, approximately linear relations observed between sets of masses and spins of particles otherwise having the same set of quantum numbers, could be explained in terms of strings rotating in flat space. This approach was however riddled with problems, such as the appearance of a massless particle of spin 2, which had not been observed experimentally, and was soon abandoned. Later it was realized that, rather than theories of the strong interactions, such string theories could provide candidate theories for a unified theory of all interactions, including gravity, where the spin 2 particle takes the role of the graviton. More than 20 years have been dedicated to explore this possibility, spurring a flurry of developments in both physics and mathematics.

In late 1997, a new application of string theory emerged, re-launching it as a theory of the strong interactions. Based on ideas put forward earlier by 't Hooft on holography [1], i.e. the dual description of reality in terms of a theory of a different space-time dimension, and properties of superstring theory, Juan Maldacena conjectured [2] that $\mathcal{N} = 4$ Super Yang Mills has a dual description in terms of the ten-dimensional type IIB superstring theory, living in the curved space-time $AdS_5 \times S^5$. One of the main arguments in favor of this duality was precisely that the symmetry groups of the two theories match. This gauge/string duality came to be known as the AdS/CFT correspondence, and Maldacena's article has become one of the most cited papers of high energy physics. Further developed in [3] and [4], the correspondence, and its generalizations, has found many applications, in for example quark/gluon plasma physics and superconductivity.

The main strength of AdS/CFT is that it provides a dual description of strongly coupled gauge theory in terms of weakly coupled, and thus perturbatively accessible, string theory. More precisely, with gauge group $SU(N)$,

$$\frac{4\pi\lambda}{N} = g_s \ , \quad \sqrt{\lambda} = \frac{R^2}{\alpha'} \ , \tag{1.0.1}$$

where $\lambda \equiv g_{YM}^2 N$ is the 't Hooft coupling, g_s the string coupling, R the common radius of AdS_5 and S^5, and α' is the string slope. The second of these relations implies that the gauge theory is weakly coupled when the string theory background is highly curved, i.e. strongly coupled, and vice versa. The difference between the original attempt to describe the strong interactions in terms of string theory and AdS/CFT is thus, besides the AdS background, that the weakly coupled degrees of freedom of the gauge and string theory do not occupy the same region of validity. Among other things, such as the space-time dimension, this reconciles the appearance of the graviton in the spectrum. In fact, Brower et al [5] have shown that the weak coupling Regge trajectory corresponding to an exchange of singlet quantum numbers, containing the perturbative pomeron, to be explained in section 5.1.3, takes the form of a graviton Regge trajectory at strong coupling.

So the applicability of AdS/CFT lies in the weak/strong coupling map (1.0.1), but this same relation also makes it very difficult to prove the correspondence, since information about either theory beyond their perturbative regimes is limited. Traditionally, lacking a proof one has instead tried to give as much weight to the correspondence as possible, by calculating observables in both theories and trying to find agreement.

Two classes of observables in which this has been fruitful is the spectrum of anomalous dimensions, discussed in chapter 2, and the calculation of scattering amplitudes, reviewed in chapter 4. For the most part, these two classes of observables live in different worlds, but in some cases they have been able to make contact. The cusp anomalous dimension [6] is an important component of the four-particle and five-particle amplitudes [7], and its first four-loop evaluation was indeed performed within this framework [8]. A more surprising connection, mentioned in section 2.5.1, is that the Regge limit of scattering amplitudes [9], and the BFKL equation [10], the topic of chapter 5, seems to contain the same information as an analytical continuation of the anomalous dimensions of so-called quasi-partonic twist operators [11]. Historically, this relation was exploited in [12] to show that the model used at the time to calculate anomalous dimensions failed starting from four loops, and is still used in recent articles, such as [13], as an important constraint for newer proposals for the spectrum of anomalous dimensions.

Excitingly, in recent years it has been discovered that both classes of observables exhibit hidden, infinite dimensional symmetry algebras, characteristic of integrability. This symmetry enhancement appears in the limit $N \to \infty$, known as the planar limit. The name stems, as explained by 't Hooft [1], from the dominance of those Feynman diagrams that can be drawn on a two-dimensional surface in this limit. This opens the possibility that $\mathcal{N} = 4$ SYM and type IIB string theory on $AdS_5 \times S^5$ be exactly solvable, at least in the planar limit, allowing for a proof of planar AdS/CFT.

Symmetry thus plays a crucial role in this context as well, and this thesis is precisely concerned with two conjectured symmetries, each pertaining to one of the classes of observables. The first symmetry appears in the spectrum of anomalous dimensions when embedding $\mathcal{N} = 4$ into the so-called marginally β-deformed theory [14], which reduces to $\mathcal{N} = 4$ when $\beta = 0$, and is the topic of chapter 3. Evidence for this symmetry, relating anomalous dimensions at different values of β, and which can be used to impose constraints on the undeformed theory, is produced by analytically continuing the spectrum of the original theory in a way that is explained in section 2.5.2. The second symmetry, discussed in section 5.3, is a property of the BFKL equation, and is suggested to be a high energy remnant of the corresponding symmetry, known as dual conformal symmetry [15], of scattering amplitudes. This latter relation has the potential to explain why integrability is found in the high energy limit of gauge theories [16].

The work presented is based on the papers

1. C. Gomez, J. Gunnesson and R. Hernandez, *"Magnons and BFKL"*, JHEP **0809**, 060 (2008), `arXiv:0807.2339`.

2. J. Gunnesson, *"Wrapping in maximally supersymmetric and marginally deformed N=4 Yang-Mills"*, JHEP **0904**, 130 (2009), `arXiv:0902.1427`.

3. C. Gomez, J. Gunnesson and A. S. Vera, *"Dual conformal invariance in the Regge limit"*, Phys. Lett. B **690** (2010) 78, `arXiv:0908.2568`.

4. J. Gunnesson, *"Commuting Conformal and Dual Conformal Symmetries in the Regge limit"*, `arXiv:1003.4193`.

while the early paper

C. Gomez, J. Gunnesson and R. Hernandez, *"The Ising model and planar N=4 Yang-Mills"*, J. Phys. A **41** (2008) 275205, `arXiv:0711.3404`.

is mentioned briefly.

Chapter 2

The spectrum of anomalous dimensions and Integrability

In this chapter we will discuss the developments in the calculation of the spectrum of anomalous dimensions of gauge-invariant operators, on the gauge theory side, and, in a briefer fashion, the spectrum of energies on the string theory side. If AdS/CFT is correct the two spectra should coincide, but due to the non-overlap of their perturbative regimes direct checks are difficult to carry out directly. Over the past few years the problem seems to be headed towards a solution, as input from both sides of the correspondence have allowed the construction of an integrable model, reproducing all known data on the AdS/CFT spectrum. The chapter contains some of the authors work, in section 2.5.2, but is otherwise mainly a review of the literature. Recent works also containing reviews of this topic are [19], while the earlier treatments [20, 21] are also recommended.

In the next section the problem of comparing the spectra on both sides of the AdS/CFT correspondence will be explained in more detail. The developments leading up to its possible solution will then be briefly described. Section 2.2 contains early developments on the subject, including the BMN limit, followed by the complete determination of the one-loop spectrum, and the appearance of integrability, in section 2.3. Section 2.4 then describes the construction, and evidence for, an all-loop Asymptotic Bethe Ansatz, giving the anomalous dimensions for a large class of operators, with section 2.5 studying the $SL(2)$ sector of operators in greater detail. Section 2.6 then contains the final piece of the puzzle, incorporating wrapping effects into the model. Finally, in 2.7, we reflect on how the current models have emerged through an interaction of gauge and string theoretic considerations.

2.1 $\mathcal{N} = 4$ Super Yang Mills and the spectrum of anomalous dimensions

The four-dimensional gauge theory involved in the original, and best understood, version of AdS/CFT is $\mathcal{N} = 4$ Super Yang Mills, the maximally supersymmetric extension of pure Yang-Mills theory. Formulated in [17], this theory was shown in [18] to have the important property of conformal invariance at the quantum level. The β function of the theory thus vanishes to all orders in perturbation theory.

The particle content of the theory, which is determined by the requirement of $\mathcal{N} = 4$ supersymmetry in four dimensions, is one gauge field A_μ, four Majorana fermions $\psi_{\alpha a}$, $\dot{\psi}^a_{\dot{\alpha}}$, and six scalars ϕ_m. Here, μ is a space-time index taking values $0, \ldots, 3$, α and $\dot{\alpha}$ are $su(2)$ indeces corresponding to the Lorentz group, a takes values from 1 to 4, while m goes from 1 to 6. These last two indices reflect that the fermions transform in the fundamental representation of the R-symmetry $su(4) \cong so(6)$, while the scalars transform in the vector representation. All of the fields transform in the adjoint representation of the gauge group, which we will take to be $SU(N)$.

Introducing the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - i g_{YM} \left[A_\mu,\, \cdot\,\right] \ , \tag{2.1.1}$$

where g_{YM} is the gauge coupling, and the field strength

$$\mathcal{F}_{\mu\nu} = i g_{YM}^{-1} \left[\mathcal{D}_\mu,\, \mathcal{D}_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_{YM} \left[A_\mu,\, A_\nu\right] \ , \tag{2.1.2}$$

we can write the Lagrangian as

$$\begin{aligned} \mathcal{L} = &\frac{1}{4}\mathrm{Tr}\, \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\mathrm{Tr}\, \mathcal{D}^\mu \phi^n \mathcal{D}_\mu \phi_n + \mathrm{Tr}\, \dot{\psi}^a_{\dot{\alpha}} \sigma_\mu^{\dot{\alpha}\beta} \mathcal{D}^\mu \psi_{\beta a} - \\ &- \frac{1}{2} i g \mathrm{Tr}\, \psi_{\alpha a} \sigma_m^{ab} \varepsilon^{\alpha\beta} \left[\phi^m,\, \psi_{\beta b}\right] - \frac{1}{2} i g \mathrm{Tr}\, \dot{\psi}^a_{\dot{\alpha}} \sigma^m_{ab} \varepsilon^{\dot{\alpha}\dot{\beta}} \left[\phi^m,\, \dot{\psi}^b_{\dot{\beta}}\right] \ , \end{aligned} \tag{2.1.3}$$

where ε is the usual completely anti-symmetric tensor, the $\sigma_\mu^{\dot{\alpha}\beta}$ are given by the Pauli matrices, while the σ_m^{ab} are their six-dimensional analogues (the chiral projections of the six-dimensional γ-matrices).

The local symmetry of the Lagrangian is $PSU(2,2|4)$, which coincides with the symmetry of the superstring σ-model defined on $AdS_5 \times S^5$ (we will present the algebra in more detail in section 4.2, in the context of scattering amplitudes). In particular, the ordinary conformal group $SO(4,2)$ corresponds precisely to the isometry group of the AdS_5 space. In this relation the dilatation generator D of $\mathcal{N} = 4$, which generates radial dilatations, is mapped to the Hamiltonian of the string theory, responsible for translations in the global

AdS time. If AdS/CFT is correct, the eigenvalues of the dilatation operator should coincide with the spectrum of energies of the string theory. A priori, it is however not clear how to perform the map that relates string states to operators of the gauge theory.

If a set of local operators $\mathcal{O}_m(x)$ are eigen-operators of the dilatation operator, they satisfy the commutation relation

$$[D,\, \mathcal{O}_m(x)] = \Delta_m \mathcal{O}_m(x) \ , \tag{2.1.4}$$

where Δ_m is called the scaling dimension of the operator. The scaling dimension decomposes as

$$\Delta_m = \Delta_m^0 + \gamma_m \ , \tag{2.1.5}$$

where Δ_m^0 is the classical dimension of the operator, obtained by simply counting the dimensions of its constituent fields, while γ_m is the anomalous dimension, which acquires a non-zero value due to quantum effects. Despite $\mathcal{N} = 4$ being a UV finite theory, the operators $\mathcal{O}_m(x)$ will introduce new, short-distance singularities, and must still be renormalized. It is this process of renormalization that gives rise to the anomalous dimensions.

The set of operators $\mathcal{O}_m$ will have a two-point correlation function given by

$$\langle \mathcal{O}_m(x) \mathcal{O}_n(y) \rangle = \frac{\text{const}\ \delta_{mn}}{|x-y|^{2\Delta_m}} \ , \tag{2.1.6}$$

so in order to calculate the spectrum of anomalous dimensions, one can evaluate such two-point correlators, a method which was used frequently in the early days of the subject. The difficulty lies in that a given operator will usually not be an eigen-operator of the dilatation operator, or equivalently, it will mix with other operators under renormalization. In order to calculate the spectrum in this framework, one must therefore solve the problem of operator mixing.

As mentioned in the introduction, we will almost exclusively be interested in the planar limit, $N \to \infty$, where only those Feynman diagrams that can be drawn on a two-dimensional surface will be relevant. As explained by 't Hooft, the relevant coupling parameter in this limit is the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$. Following a frequent convention of the literature, we will use the following re-scaled version of the 't Hooft coupling

$$g^2 \equiv \frac{\lambda}{16\pi^2} \ . \tag{2.1.7}$$

Most of the progress on the calculation of anomalous dimensions has been done in the planar limit. Then, gauge-invariant operators with a different number of color traces do not mix under renormalization. It is therefore sufficient to restrict attention to local operators of the form

$$\mathcal{O}_m(x) = \mathrm{Tr}\left(\mathcal{X}_1(x) \cdots \mathcal{X}_L(x)\right) \ , \tag{2.1.8}$$

with Tr being a color trace and where the $\mathcal{X}_i$ can be any of the fundamental scalar or fermion fields, together with covariant derivatives acting on such fundamental fields. It is clear that the operators (2.1.8) must be renormalized since the $\mathcal{X}_i$ act on the same space-time point x.

2.2 Early checks and BMN

Since the perturbative regimes of the gauge and string theories do not coincide direct comparisons of the spectra are difficult. Early on, Witten showed in [4] that the spectrum of chiral operators, which are protected by supersymmetry and thus have the same scaling dimensions for all values of the coupling, coincides with the energies of Kaluza-Klein modes in the supergravity approximation (corresponding to $\lambda \to \infty$) of the string theory. But since the global symmetry algebras of the two theories coincide one would like to perform comparisons not protected by SUSY.

Such a non-trivial check was performed in [22] by Berenstein, Maldacena and Nastase. By taking a special limit of the string theory, known as the plane-wave limit, in which it becomes possible to quantize the theory exactly, they could identify which gauge theory operators were dual to the string states in this limit and compare the spectra.

The plane wave limit consists of letting a string, degenerate to a single point, move very fast along the S^5. With J being its angular momentum on the S^5, one takes the limit

$$J \sim R^2 \sim \sqrt{N} \to \infty \ , \tag{2.2.1}$$

where R again is the radius of $AdS_5 \times S^5$ and N the number of colors of the gauge symmetry. Considering fluctuations around this configuration, the string effectively lives in a plane wave, or pp wave geometry, in which the σ-model Lagrangian becomes quadratic and one can perform an exact quantization of the theory. Quantizing by expanding in Fourier modes, the energy $E = \Delta$ of an arbitrary string state is given by

$$\Delta - J = \sum_{n=-\infty}^{\infty} N_n \sqrt{1 + \frac{\lambda n^2}{J^2}} \ , \tag{2.2.2}$$

where N_n is the occupation number for mode n, and where one has the constraint

$$P = \sum_{n=-\infty}^{\infty} n N_n = 0 \ . \tag{2.2.3}$$

corresponding to the level-matching condition, or equivalently, to the vanishing of the total world-sheet momentum. The contribution from mode number n is therefore

$$(\Delta - J)_n = \sqrt{1 + \frac{\lambda n^2}{J^2}} \ . \tag{2.2.4}$$

The AdS/CFT relations (1.0.1) imply that the BMN limit (2.2.1) corresponds to taking g_{YM} fixed and $N \to \infty$ in the gauge theory. Since quantum corrections appears multiplied by $\lambda = g_{YM}^2 N$ it would thus seem that this limit is outside of the perturbative region of $\mathcal{N} = 4$ SYM. However, the key point of [22] was that for certain operators close to the protected, chiral operator $\mathrm{Tr} Z^J$, where $Z = \phi^5 + i\phi^6$, the factor multiplying quantum corrections is

$$\lambda' \equiv \frac{\lambda}{J^2} \,, \tag{2.2.5}$$

which can be small in the BMN limit.

The way to reproduce the formula (2.2.2) in the gauge theory is to start by identifying the operator $\mathrm{Tr} Z^J$ with the string state having all modes unoccupied. This identification follows from it being the unique operator having scaling dimension $\Delta = J$. Berenstein, Maldacena and Nastase then explained that occupying modes in the string state corresponds to adding impurities, such as scalars ϕ^i, in the operator $\mathrm{Tr} Z^J$, introduced with a position dependent phase. If one introduces a ϕ^i-field, for example, the operator corresponding to mode number n is

$$\sum_{l=1}^{J} \mathrm{Tr}\left[Z^l \phi^i Z^{J-l}\right] e^{\frac{2\pi i n l}{J}} \ . \tag{2.2.6}$$

This operator vanishes by the cyclicity of the trace, or the constraint (2.2.3) for the string, but one can continue inserting several fields, accompanied by similar phases, in such a way that the total momentum is zero. Operators constructed in this way are known as BMN operators. Often, the term refers to the special case of two impurities, which for the case of scalar impurities takes the form

$$\sum_{l=1}^{J} \mathrm{Tr}\left[\phi^i Z^l \phi^j Z^{J-l}\right] e^{\frac{2\pi i n l}{J}} \ . \tag{2.2.7}$$

Expanding (2.2.2) in the coupling, it was checked in [22] that one obtains precisely the anomalous dimensions of such scalar operators at one loop.

The pp wave string/BMN operator-correspondence was then strengthened in subsequent work. In [23] a two-loop calculation of BMN operator dimensions was performed, and an all-loop resummation was performed, under some assumptions. The all-loop formula (2.2.2) was then shown algebraically to all orders in the gauge theory in [24]. Evidence was also given for the universality of (2.2.4), i.e. its independence with respect to the excitation type, in for example [25] and [26], where the same anomalous dimensions are obtained if one, or both, scalar impurities are exchanged for covariant derivatives. Evidence for this property, when non-planar corrections are taken into account, was also given in [27]. This universality of BMN operator dimensions seemed surprising at the time, from the point of view of the gauge theory, since different such operators belong to different supermultiplets, and will indeed have different anomalous dimensions for finite J [28].

2.3 One-loop integrability.

The success of BMN motivated the calculation of anomalous dimensions outside of this limit. One must then take into consideration operator mixing, which had been studied earlier, but it was with the famous calculation of Minahan and Zarembo in [29] that things started to get interesting. They discovered that the problem of calculating operator dimensions, at one-loop in the planar limit, for single trace operators of scalar fields, could be mapped to the problem of diagonalizing an integrable spin chain! Integrability implies the presence of an infinite amount of symmetry, allowing powerful techniques, such as the Bethe Ansatz, for calculating the spectrum. It was then noticed that the problem could be further simplified by focusing on the dilatation operator [30], and the one-loop integrable spin chain was subsequently extended to the entire algebra [31, 32].

2.3.1 The $SO(6)$ spin chain and the Bethe Ansatz.

As mentioned earlier, the calculation of (2.1.6) can give the anomalous dimensions for operators of a definite conformal dimension. In evaluating the relevant loop integrals, ultraviolet divergences arise, as usual. This is not a disadvantage, though, since introducing a hard cutoff Λ, one obtains a convenient expression for the anomalous dimensions. A dimensional analysis tells us that with bare, unrenormalized operators on the left hand side of (2.1.6) and with $m = n$, this side has twice the classical dimension of $\mathcal{O}^m$. Assuming that (2.1.6) has been diagonalized, the right hand side has, besides the classical dimension, an anomalous part, with mass dimension $2\gamma_m$. For dimensional reasons, we must therefore insert a factor of $\Lambda^{-2\gamma_m}$ into the right hand side. But if we require this side to be independent of the cut-off we must renormalize the operators in order to absorb this Λ dependence. We are thus prompted to define renormalized operators

$$\mathcal{O}^m_{\text{ren}} = Z^m \mathcal{O}^m \ , \tag{2.3.1}$$

where $\mathcal{Z}^m$ is Λ^{γ_m}. We can then extract the anomalous dimension as

$$\gamma_m = (Z^m)^{-1} \frac{dZ^m}{d\log\Lambda} \ . \tag{2.3.2}$$

In the general case, when (2.1.6) has not been diagonalized, Z will be a non-diagonal matrix $Z^m{}_n$, describing how the bare operators mix among themselves under renormalization, through

$$\mathcal{O}^m_{\text{ren}} = Z^m{}_n \mathcal{O}^n \ . \tag{2.3.3}$$

The renormalization factor Z is determined by requiring finiteness of the correlator

$$\left\langle Z_\Phi^{1/2}\Phi_{j_1}(x_1)\cdots Z_\Phi^{1/2}\Phi_{j_L}(x_L)\mathcal{O}^m_{\text{ren}}\right\rangle \ , \tag{2.3.4}$$

where Z_Φ is the wave-function renormalization factor chosen so that the two-point correlator $\langle \Phi_i \Phi_j \rangle$ be finite. The mixing matrix is then given by

$$\Gamma = Z^{-1} \frac{dZ}{d \ln \Lambda} \ , \tag{2.3.5}$$

and its diagonalization will provide the spectrum of anomalous dimensions.

In [29] Minahan and Zarembo calculated the one-loop mixing matrix, equivalent the one-loop dilatation operator, for generic scalar operators

$$\mathcal{O}^m = a^{i_1 \ldots i_L} \mathrm{tr} \Phi_{i_1} \cdots \Phi_{i_L} \ , \tag{2.3.6}$$

with the $a^{i_1 \ldots i_L}$ being coefficients. At one-loop, these scalar operators will only mix among themselves, and only operators of the same length L will mix.

A key observation made by Minahan and Zarembo is that we can map the set of operators (2.3.6) into the states of a periodic (due to the trace) $SO(6)$ spin chain of length L, where each of the scalars sits at a site of the chain. The symmetry of the chain is $SO(6)$ since this is the R-symmetry of the $\mathcal{N} = 4$ theory, which rotates the scalars among themselves in the vector representation. The matrix Γ can then be interpreted as the Hamiltonian for this spin chain, and the problem of finding its spectrum is equivalent to diagonalizing this Hamiltonian. Furthermore, a consequence of taking the planar limit is that only neighboring fields in the color trace (2.3.6) can interact, at one loop, implying the Hamiltonian has only nearest-neighbor interactions. All states of the spin chain do not correspond to an operator, however. Since the trace implies translational invariance one must also impose an additional, zero momentum constraint, restricting the spin chain.

Labeling the sites of the spin chain as $i = 1, \ldots, L$, calculating the relevant Feynman diagrams, Γ, or equivalently, the one-loop dilatation operator is obtained as

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{l=1}^{L} \left(K_{l,l+1} + 2 - 2P_{l,l+1} \right) \ , \tag{2.3.7}$$

where P is the permutation operator and where the trace operator K acts as

$$K \, a \otimes b = a \cdot b \sum_i \hat{e}^i \otimes \hat{e}^i \ , \tag{2.3.8}$$

where the $\hat{e}^i$ constitute an orthonormal basis of $\mathbb{R}^6$. Apart from the notational simplicity, the reason for introducing the spin chain language is that the ratio between the coefficients of K and P in (2.3.7) turns out to be precisely -2, which Minahan and Zarembo explain is required for the spin chain to be integrable, meaning that it has a number of conserved charges equal to the degrees of freedom of the system. The powerful techniques of integrability can therefore be used to diagonalize Γ and thereby obtain the one-loop spectrum of anomalous dimensions for the scalar single trace operators.

A technique especially well adapted to the $SO(6)$ spin chain is the Bethe Ansatz, which we now explain for the simpler case of the $SU(2)$ subsector. This subsector consists of single-trace operators of the complex scalar fields $Z = \phi^5 + i\phi^6$ and $W = \phi^3 + i\phi^4$, which thus have the form

$$\mathrm{Tr}\,[ZZWZWZ\cdots ZZ] \ . \tag{2.3.9}$$

It is called the $SU(2)$ sector because it preserves an $SU(2)$ subgroup of the R-symmetry, under which the Z and W transform in the fundamental representation. A key property of this sector is that it is closed under operator mixing to all orders in perturbation theory, since there are no operators outside this sector having the same classical dimension and same R-charges as a given operator of the form (2.3.9). The spin chain map is also particularly simple to visualize in this case since one can take, for example, the Z fields to be mapped to spin "up" and the W field to spin "down".

For the operators (2.3.9) the trace operator K vanishes, leaving a Hamiltonian

$$H = \sum_{l=1}^{L} (1 - P_{l,l+1}) \ , \tag{2.3.10}$$

where we have extracted the factor $\frac{\lambda}{8\pi^2}$ from Γ. Curiously, this coincides with the Hamiltonian for the Heisenberg XXX spin chain, which was the first model to be solved, by Hans Bethe in [33], by the Bethe Ansatz.

In applying the coordinate Bethe Ansatz[1] one starts, just as in the construction of the BMN operators, with the operator TrZ^L, interpreted as a ferromagnetic ground state $|\ \rangle$ of the spin chain, and then adds additional fields in a type of plane-wave Ansatz. The momenta of these plane waves, and their relative coefficients are determined by a set of equations called the Bethe equations.

Let us define the spin-flipped state $|j\rangle$ as the state obtained from $|\ \rangle$ by flipping the spin at site j, corresponding to exchanging a Z field for a W field at that position. We then define a single "magnon" state of momentum p by

$$\sum_{j}^{L} e^{ipj}|j\rangle \ . \tag{2.3.11}$$

Imposing the zero-momentum condition implies that $p = 0$, leaving only the protected state corresponding to

$$\mathrm{Tr}\,[WZZ\cdots Z] \ , \tag{2.3.12}$$

but if we ignore this for the moment and act on (2.3.11) with (2.3.10) we find that this is an eigenstate of the Hamiltonian and obtain the magnon energy

$$E(p) = 4\sin^2\frac{p}{2} \ , \tag{2.3.13}$$

[1]In contrast with the algebraic Bethe Ansatz, reviewed in for example [34].

giving the energy of the one-magnon state.

We can then move on to multiple spin flips, defining $|j,\, k\rangle$, with $j < k$, as the state obtained from $|\rangle$ by flipping the spins at sites j and k. The Bethe Ansatz consists in looking for two-magnon states of the form

$$|p_1,\, p_2\rangle \equiv \sum_{j<k} \left(e^{ip_1 j + ip_2 k} + S(p_1,\, p_2) e^{ip_1 k + ip_2 j}\right) |j,\, k\rangle \ , \tag{2.3.14}$$

where the coefficient $S(p_1,\, p_2)$ is interpreted as an "S-matrix" since it multiples the term exchanging the momenta of the two magnons. Requiring that (2.3.14) be an eigenstate of (2.3.10) determines the S-matrix and implies that the allowed momenta p_1 and p_2 must satisfy the Bethe equation

$$e^{ip_1 L} = S(p_1,\, p_2) \ . \tag{2.3.15}$$

Together with the zero-momentum constraint $p_1 = -p_2$ this will determine a discrete set of momenta.

It turns out to be convenient to express this problem in terms of the magnon rapidity u, defined as $u \equiv \frac{1}{2} \cot \frac{p}{2}$. Inverting, this gives the momentum as

$$e^{ip} = \frac{u + i/2}{u - i/2} \tag{2.3.16}$$

and the one-loop dispersion relation (2.3.13) as

$$E(u) = \frac{1}{u^2 + \frac{1}{4}} \ , \tag{2.3.17}$$

while one finds that the magnon S-matrix takes the form

$$S(u_k,\, u_j) = \frac{u_k - u_j + i}{u_k - u_j - i} \ . \tag{2.3.18}$$

Since $u_2 = -u_1$ by the zero-momentum condition the Bethe equation (2.3.15) becomes

$$\left(\frac{u_1 + i/2}{u_1 - i/2}\right)^L = \frac{2u_1 + i}{2u_1 - i} \ . \tag{2.3.19}$$

After having solved this equation, the energy of the corresponding state is given simply by the sum of the energies of the individual magnons, i.e. $E(u_1) + E(u_2) = 2E(u_1)$.

One can then continue adding even more magnons, as long as $M \leq L$ in an analogous fashion, as products of plane waves with unknown coefficients and momenta. Fortunately, integrability simplifies this problem, implying that the multi-magnon S-matrices can be factorized completely in terms of the two-magnon S-matrix. For example,

$$S(p_1,\, p_2,\, p_3) = S(p_1,\, p_2) S(p_1,\, p_3) S(p_2,\, p_3) \ . \tag{2.3.20}$$

The Bethe equations an M magnon state take the form

$$e^{ip_k L} = \prod_{j \neq k}^{M} S(p_k,\, p_j) \ . \tag{2.3.21}$$

They can be interpreted as periodicity conditions for the magnon wavefunctions. If the magnons were non-interacting, one would need to impose the plane-wave quantization condition $e^{ip_k L} = 1$, while the Bethe equations take into consideration that the magnons acquire a phase shift, given by the S-matrix, when changing order with each other.

In terms of rapidities the equations take the form

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j \neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \ , \tag{2.3.22}$$

while the zero-momentum, or cyclicity, constraint becomes

$$\prod_k \frac{u_k + i/2}{u_k - i/2} = 1 \ . \tag{2.3.23}$$

The energy of a state corresponding to a set of rapidities $\{u_k\}$ that satisfy equation (2.3.22), known as a set of Bethe roots, is then simply given in terms of one-magnon energies as

$$E(u_1,\, \ldots,\, u_M) = \sum_k^M E(u_k) = \sum_k^M \frac{1}{u_k^2 + \frac{1}{4}} \ . \tag{2.3.24}$$

In this way one can thus obtain the spectrum of one-loop anomalous dimensions for the $SU(2)$ sector in an efficient way.

For the full $SO(6)$ spin chain, the same technique works, although it becomes a bit more complicated. There are then three different types of Bethe roots $u_{q,\,k}$, with $q = 1,\, 2$ and 3, one for each simple root of $SO(6)$, corresponding to the different types of impurities that can be inserted in $\mathrm{Tr}Z^L$ state.

Minahan and Zarembo demonstrated how one can easily calculate several anomalous dimensions using the Bethe Ansatz, reproducing, for example, the finite J BMN operator dimensions of [28]. In a later article [35], written in collaboration with Beisert and Staudacher, they used this formalism to provide the strongest quantitative check at that time of the AdS/CFT correspondence, reproducing at one loop the scaling dimensions corresponding to a family of string theory solutions [36, 37], calculated by Frolov and Tseytlin, in which two different R-charges were taken to be large. The result was highly non-trivial, being obtained by inverting certain elliptic functions, and would probably be very difficult to obtain without the Bethe Ansatz technique.

2.3.2 The dilatation operator and the full one-loop Bethe Ansatz

The one-loop $SO(6)$ mixing matrix (2.3.7) of Minahan and Zarembo coincides with the one-loop dilatation operator, restricted to scalar operators in the planar limit. In [30], following earlier insights in [38] and [39], a systematic study of the dilatation operator was initiated, and its complete one-loop construction was concluded in [31]. The main point of focusing on the dilatation operator is that its determination is heavily simplified, compared to directly evaluating the correlators (2.1.6). In fact, it can to a large extent be determined by symmetry requirements, and known results such as BMN operator dimensions. Furthermore, at one loop, the dilatation operator constructed in this way applies equally well to the non-planar case, providing insight into the mixing of, for example, single and double trace operators.

The algebraic constraints, explained in [31] lie in the following argument: Since the representation of the dilatation operator $D(\lambda)$ receives quantum corrections, it would seem natural that the other generators also do so. The algebra itself should not be deformed, however, so writing the quantum deformed generators as

$$J^a(\lambda) = J_0^a + \sum_{l=1}^{\infty} J_{2l}^a \left(\frac{g_{YM}^2}{8\pi^2}\right)^l , \tag{2.3.25}$$

where the J_0^a are the classical generators, one has

$$[D(\lambda),\, J^a(\lambda)] = \mathrm{eng}\,(J_0^a)\, J^a(\lambda) , \tag{2.3.26}$$

where $\mathrm{eng}\,(J_0^a)$ is the engineering dimension of the corresponding generator. Expanding, and canceling the lowest order terms, one then gets, to one-loop order, that

$$[D_2,\, J_0^a] + [D_0,\, J_2^a] = \mathrm{eng}\,(J_0^a)\, J_2^a . \tag{2.3.27}$$

Since the J^a only mix in perturbation theory with generators of the same classical dimension,

$$[D_0,\, J_2^a] = \mathrm{eng}\,(J_0^a)\, J_2^a \tag{2.3.28}$$

and it must be that

$$[D_2,\, J_0^a] = 0 . \tag{2.3.29}$$

The one-loop correction to the dilatation operator is thus invariant under the classical algebra.

Applying symmetry and representation theory arguments Beisert determines that, acting on two fields, the one-loop dilatation operator is given by

$$H_{12} = 2h(J_{12}) \equiv \sum_{j=0}^{\infty} 2h(j) P_{12,j} , \tag{2.3.30}$$

where $P_{12,j}$ projects the spins at sites 1 and 2 into the module V_j of spin j, and the $h(j)$ are coefficients. In [31] these coefficients are determined by comparing with the known results of [11, 40] in the $SL(2)$-sector (the topic of section 2.5) to be

$$h(j) = S_1(j) \equiv \sum_{k=1}^{n} \frac{1}{k} = \psi(j+1) - \psi(1) \ , \tag{2.3.31}$$

where $S_1(j)$ is called a harmonic sum, and $\psi(j)$ is the digamma function.

For special cases, such as the $SO(6)$-sector, more convenient representations of the dilatation operator exist. One representation, which due to its simple generalization to the marginally deformed theory will interest us the most, is given in terms of nearest neighbor permutations and is valid in the planar limit of the $SU(2)$ sector. Introducing the notation

$$\{n_1,\, n_2,\, \ldots,\, n_p\} = \sum_{k=1}^{L} P_{k+n_1,k+n_1+1} P_{k+n_2,k+n_2+1} \cdots P_{k+n_p,k+n_p+1} \ , \tag{2.3.32}$$

and identifying $\{\}$ with the identity operator, provides a convenient basis for the dilatation operator, as well as the higher conserved charges in this sector. At one-loop, as can be read off (2.3.10), we have

$$D_2 = (\{\} - \{0\}) \ , \tag{2.3.33}$$

while the next conserved charge, acting on three neighboring spins, is given by

$$U_2 = 2\left(\{1,\, 0\} - \{0,\, 1\}\right) \ . \tag{2.3.34}$$

Finally, it was shown by Beisert and Staudacher in [32], that the full, planar one-loop dilatation operator was integrable, by deriving an R-matrix. Drawing from work done on integrable structures found in QCD [41] and their relation to $\mathcal{N} = 4$ SYM, the known R-matrix of the $SL(2)$ sector was generalized to the full superalgebra, analogously to the lifting of the dilatation operator to the full algebra in [31], assuming that the resulting R-matrix be unique.

Furthermore, the dilatation operator could again be diagonalized by applying a Bethe Ansatz, preserving the full $PSU(2,2|4)$ algebra. Interestingly, this Bethe Ansatz can be obtained by generalizing the Heisenberg XXX chain equations (2.3.22)-(2.3.24) of the $SU(2)$ sector, by using a method already present in the integrability literature [43], allowing for a general algebra and representation. Associating each Bethe root u_j to a Dynkin diagram node, the Bethe Ansatz equations for $j = 1, \cdots, M$ are given by

$$\left(\frac{u_j + \frac{i}{2} V_{k_j}}{u_j - \frac{i}{2} V_{k_j}}\right)^L = \prod_{l \neq j}^{n} \frac{u_j - u_l + \frac{i}{2} M_{k_j,\, k_l}}{u_j - u_l - \frac{i}{2} M_{k_j,\, k_l}} \ , \tag{2.3.35}$$

where k_j number the Dynkin node to which the root u_j is associated, V_{k_j} is the corresponding Dynkin labels, and M_{kl} is the Cartan matrix of the algebra. The energy of the states corresponding to solutions of these equations are then given by

$$E = \pm \sum_{j=1}^{n} \left(\frac{i}{u_j + \frac{i}{2}V_{k_j}} - \frac{i}{u_j - \frac{i}{2}V_{k_j}} \right) \,, \tag{2.3.36}$$

while states corresponding to gauge theory operators must satisfy the cyclicity constraint

$$1 = \prod_{j=1}^{n} \frac{u_j + \frac{i}{2}V_{k_j}}{u_j - \frac{i}{2}V_{k_j}} \,. \tag{2.3.37}$$

Beisert and Staudacher identifed in which representation the fundamental $\mathcal{N} = 4$ fields transform, and resolved some subtleties regarding the Dynkin diagram[2] for the superalgebra so that this method could be applied.

2.4 Higher loops and the ABA

Even though the one-loop integrability of $\mathcal{N} = 4$ Super Yang Mills was encouraging, it is not as special as one would think, since gauge theories tend to have integrable sectors at one-loop[3], or at leading order of logarithmic resummations[4]. It is true that, as a four-dimensional gauge theory, $\mathcal{N} = 4$ is (together with the special class of marginal deformations of the theory, discussed in chapter 3) unique in that it has full one-loop integrability, but the important question is whether integrability persists to higher orders.

Applying AdS/CFT, the first indication that this might be the case came from the supergravity calculation of [46], where an infinite number of conserved charges were found in the $\lambda \to \infty$ limit. Shortly thereafter, it was shown in [47] that the 2d sigma-model construction of the $AdS_5 \times S^5$ superstring of [48] also presents an infinite set of nonlocal conserved charges. This result depends heavily on the fact that the target space of the sigma model is the coset

$$\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \,. \tag{2.4.1}$$

[2]In the case of superalgebras, there are several choices for the Dynkin diagrams. Different choices will lead to the same spectrum, but the Bethe Ansatz may be more convenient in some cases since the ground states will be different.

[3]Integrable spin chains appearing at one-loop in QCD are studied in [44], while the relation between the integrable subsectors of several gauge theories is given in [45].

[4]In section 5.2 the integrable spin chain of the Generalized Leading Logarithmic Approximation of high energy QCD is presented, while the integrability of the evolution equations for twist 3 operators was studied in [41].

The same tower of conserved charges was then obtained in [49] for the pure spinor superstring formulation, and was later argued to be preserved at the quantum level to all orders in [50].

On the side of the gauge theory, signs of higher loop integrability were presented by Beisert, Kristjansen and Staudacher in [30]. Going beyond the one-loop $SU(2)$ sector dilatation operator (2.3.33), they fixed the two-loop operator as

$$D_4 = 1/2\left(-4\{\} + 6\{0\} - (\{1,\,0\} + \{0,\,1\})\right) \ , \tag{2.4.2}$$

and then noted that a two-loop third conserved charge U_4 could be determined by requiring

$$[D_4,\,U_2] + [D_2,\,U_4] = 0 \ , \quad \text{and} \quad \{P,\,U_4\} = 0 \ , \tag{2.4.3}$$

where U_2 is the one-loop charge (2.3.34) and P is the parity operator. In other words, they imposed commutation at two-loops with the dilatation operator and anti-commutation with the parity operator. The fact that a non-trivial additional conserved charge, implying an additional degeneracy in the spectrum, could be constructed was a sign that integrability persisted to two loops. This was a very interesting result at the time since not many integrable spin chains were known having Hamiltonians with non-nearest-neighbor interactions. Furthermore, assuming that the integrability, in the form of the existence of U persisted to higher loops, together with correct behavior in the BMN limit, Beisert, Kristjansen and Staudacher constructed the three-loop dilatation operator, as well as, up to two unknown constants, the four-loop operator.

As an example, this allowed the calculation of the anomalous dimension of the Konishi operator

$$\mathrm{Tr}\Phi_n\Phi_n \ , \tag{2.4.4}$$

which is the shortest non-protected operator of the theory, up to three loops since the $SU(2)$ sector operator

$$K' = \mathrm{Tr}\,[\phi,\,Z]\,[\phi,\,Z] \tag{2.4.5}$$

is a descendent of the Konishi operator and therefore has the same anomalous dimension. One obtains

$$\Delta_{K'} = 4 + 12g^2 - 48g^4 + 336g^6 + \cdots \ . \tag{2.4.6}$$

In [51] it was then shown that the assumption of three-loop integrability could be relaxed, if one considers the larger $SU(2|3)$ sector, consisting of three scalars ϕ_a, $a = 1,\,2,\,3$, and two fermions ψ_α, $\alpha = 1,\,2$, and including the $SU(2)$ sector. Algebraic constraints, Feynman diagram structure and coincidence with the BMN limit are then sufficient to determine the three-loop dilatation operator. If BMN scaling, which was not rigorously known to exist at three loops at the time, is relaxed the three-loop $SU(2|3)$ dilatation operator contains two unknown coefficients, but remains integrable. These two coefficients where later definitely fixed in [52] through the calculation of two three-loop anomalous dimensions, one of which corresponds to the Konishi operator, in accordance with (2.4.6), thus proving three loop BMN scaling.

An interesting new property appearing at higher loops in the $SU(2|3)$ sector is that the spin chain becomes dynamic, in the sense that it can fluctuate in length. The reason is that the combinations $\phi_{[1}\phi_2\phi_{3]}$ and $\psi_{[1}\psi_{2]}$ have the same quantum numbers, implying that operators with a different number of fields can mix under renormalization. This mixing also implies that the $SO(6)$ sector is no longer closed beyond one loop. It would seem that the spin chain picture would not allow for a non-definite length, reflected in that the Bethe Ansatz that Beisert constructs in [51] for the $SU(2|3)$ sector indeed has several sets of Bethe roots corresponding to different spin chain lengths. The resolution of this problem is that imposing the cyclicity constraint, the result of the spin chain being translationally invariant due to the trace, makes the different sets of Bethe roots equivalent. The cyclicity constraint thus seems to play an important role in the full theory. At one loop it only truncates the spectrum to zero-momentum states, but at higher loops it becomes a necessary consistency condition.

2.4.1 An all-loop Ansatz

The higher loop evidence presented in the previous section would seem to suggest the existence an all-loop integrable model for $\mathcal{N} = 4$. In [53], a first attempt was made in this direction with the embedding of the three-loop $SU(2)$ dilatation operator of [30] and [51] in the all-loop Inozemtsev spin chain [54]. This spin chain had several interesting properties. To begin with, the range of the interaction of its Hamiltonian increased by one site for each new perturbative order, just like the dilatation operator. This implies that the Bethe Ansatz constructed when diagonalizing the Hamiltonian is only valid up to a perturbative order equalling the length of the spin chain. It is thus an "Asymptotic Bethe Ansatz", a term which in the context of AdS/CFT first appeared in connection with the Inozemtsev chain, which simply means that its validity is restricted in this perturbative sense. The reason is that the Bethe Ansatz does not take into consideration interactions that wrap around the entire spin chain. These so-called "wrapping" contributions will be the topic of section 2.6 and chapter 3. Another interesting property of the Inozemtsev chain is that it violates BMN scaling, i.e. fails to reproduce the formula (2.2.2), starting from four loops in the weak coupling expansion, although it still seemed to have such a behavior at strong coupling. This suggests that BMN scaling might also fail at some order for the full $\mathcal{N} = 4$ theory.

At the time, however, it did not seem like a good idea to abandon the BMN results, since they seemed to provide such a nice match between gauge and string theories. In [55], properties that the dilatation operator should have, such as the strong coupling behavior of anomalous dimensions, were instead derived by imposing BMN scaling. This logic was then taken further in the famous paper [56] by Beisert, Dippel and Staudacher. In this article, a new long-range, spin chain is proposed for the $SU(2)$-sector, constructed by assuming integrability and BMN scaling, together with symmetry and field theory structure considerations. Interestingly, the spin chain so obtained is unique up to at least five loops. An all-loop proposal for a Bethe Ansatz is then given, having the correct structure in the BMN limit, and which was hoped could provide a way to obtain the exact spectrum without actually diagonalizing the dilatation operator.

The Hamiltonian, which is asymptotic, meaning that it is does not include wrapping effects and therefore valid up to the perturbative order that equals the spin chain length, was written down up to five loops in the basis of generalized permutations (2.3.32), where the five loop result was obtained in [20].

The all-loop Bethe Ansatz, satisfying BMN scaling, is obtained by generalizing the one-loop Ansatz of Minahan and Zarembo, restricted to the $SU(2)$ sector, which we, for convenience, reproduce here:

$$e^{ip_k L} = \prod_{j\neq k}^{M} \frac{u(p_k) - u(p_j) + i}{u(p_k) - u(p_j) - i} \ , \tag{2.4.7}$$

where

$$u(p) = \frac{1}{2}\cot\frac{p}{2} \ , \tag{2.4.8}$$

which can be inverted to give

$$e^{ip} = \frac{u + \frac{i}{2}}{u - \frac{i}{2}} \ . \tag{2.4.9}$$

Also, in terms of the rapidities u, the values for the conserved conserved charges of a state corresponding to Bethe roots $u_1, \ldots, u_M$ are simply given by

$$Q_r = \sum_{k}^{M} \boldsymbol{q}_r(u_k) \ , \tag{2.4.10}$$

in terms of single magnon charges

$$\boldsymbol{q}_r(u) = \frac{i}{r-1}\left(\frac{1}{\left(u+\frac{i}{2}\right)^{r-1}} - \frac{1}{\left(u-\frac{i}{2}\right)^{r-1}}\right) \ . \tag{2.4.11}$$

In particular, applying (2.4.8), the second charge, giving the magnon energies and thus, up to the constant factor $2g^2$, the anomalous dimensions, is

$$E(p) = 4\sin^2\frac{p}{2} \ . \tag{2.4.12}$$

The S-matrix, giving, as discussed in section 2.3, the factors of the right hand side of (2.4.7), of the all-loop ansatz is then obtained by simply generalizing the rapidity map (2.4.8) to

$$u(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}\ , \tag{2.4.13}$$

where we recall that $g^2 = \frac{\lambda}{16\pi^2}$. The only other ingredient that we need in order to apply the Bethe Ansatz to the computation of anomalous dimensions, the second conserved charge, is generalized to

$$E(p) = \sqrt{1+16g^2\sin^2\frac{p}{2}} - 1\ . \tag{2.4.14}$$

This gives the all-loop magnon dispersion relation. We see that setting $p = \frac{2\pi n}{J}$, and expanding the sine for large J gives precisely the BMN formula (2.2.4), up to a constant shift. There are also similar trigonometric expression for the higher charges. This Bethe Ansatz coincides with the one stemming from the Inozemtsev chain up to three loops, but differ starting from four loops in such a way that BMN scaling is preserved.

We will now introduce a new set of variables under which the expressions simplify, being rational instead of trigonometric, and which have become the standard variables for the current all-loop Bethe Ansatz. Let us define a new variable x by

$$x(u) = \frac{1}{2}u + \frac{1}{2}\sqrt{u^2-4g^2}\ , \tag{2.4.15}$$

and set

$$x^{\pm} \equiv x\left(u \pm \frac{i}{2}\right)\ . \tag{2.4.16}$$

It then turns out that the expressions become compact when rewritten in terms of the $x^{\pm}$, keeping in mind that they are related, as follows from (2.4.15) and (2.4.16), by the condition [5]

$$x^+ + \frac{g^2}{x^+} - x^- - \frac{g^2}{x^-} = i\ . \tag{2.4.17}$$

Indeed, in terms of the $x^{\pm}$ we find

$$e^{ip} = \frac{x^+}{x^-} \tag{2.4.18}$$

for the momentum and

$$\boldsymbol{q}_r = \frac{i}{r-1}\left(\frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}}\right)\ . \tag{2.4.19}$$

[5] It would be nice to have a deeper understanding of the meaning of the $x^{\pm}$ variables. In [57] we noted that these variables could be mapped in a direct way to certain transfer matrix elements of the two-dimensional Ising model, where the relation (2.4.17) defines the elliptic curve on which the Ising model couplings live. It is, however, unclear whether this map can shed any light on the $\mathcal{N}=4$ integrable model.

for the rest of the conserved charges. In particular, the magnon energy, which coincides with the second conserved charge, is given by

$$E = \frac{i}{x^+} - \frac{i}{x^-} \ . \tag{2.4.20}$$

The all-loop formulae for the conserved charges are thus obtained from the original expressions (2.4.9) and (2.4.11) by substituting

$$u \pm \frac{i}{2} \to x^{\pm} \ . \tag{2.4.21}$$

In fact, an alternative way to obtain the all-loop Ansatz from the one-loop Heisenberg chain is to perform the substitution (2.4.21) in the expressions for the conserved charges, while the S-matrix, in terms of the rapidities u_k is left untouched. The relation between the u and x variables is then given by (2.4.15), which can be solved in terms of the $x^{\pm}$ as

$$u = \frac{1}{2}\left(x^+ + \frac{g^2}{x^+} + x^- + \frac{g^2}{x^-}\right) \ . \tag{2.4.22}$$

An important property of the map (2.4.15) is that its inverse,

$$u(x) = x + \frac{g^2}{x} \tag{2.4.23}$$

is invariant under the change

$$x \leftrightarrow \frac{g^2}{x} \ , \tag{2.4.24}$$

which of course corresponds to taking a different branch of the square root in (2.4.15), so that the map between x and u is a double covering map. At weak coupling, the branch $x \approx u$ must be chosen, in order to recover the Heisenberg one-loop chain. The other branch can not be given a physical motivation since it would correspond to a negative anomalous dimensions at weak coupling. At strong coupling, however, one has the freedom to make a choice of branch, which, as pointed out in [58], resolves an apparent paradox pointed out in [59]. The problem is that the $SU(2)$ sector is closed to all orders in the weak coupling perturbative expansion, but mixes with a second $SU(2)$ factor on the string side, due to the fact that the isometry group of the three sphere S_3 is precisely $SO(4) \cong SU(2) \times SU(2)$. The solution is that the map (2.4.24) exchanges the charges of the two $SU(2)$ groups, and the second copy of $SU(2)$ is thus related to the double covering of u.

Furthermore, as noted in [56], the substitution (2.4.21) itself can be given a string theoretic motivation. Taking the thermodynamic limit, in which both L and M are taken to infinity with L/M fixed, which is a continuum limit, the magnon conserved charges become

$$\frac{\tilde{p}(u)}{x(u)^{r-1}} \ , \tag{2.4.25}$$

where $x(u)$ is given by the same map (2.4.15) as before, while

$$\tilde{p}(u) = \frac{1}{\sqrt{u^2 - 4g^2}} \ , \tag{2.4.26}$$

which coincides with the expressions for the conserved charges of the semi-classical string in the $SU(2)$ sector, as given in [60], where x plays the role of the spectral parameter and u is introduced so that one recovers the Heienberg XXX model as $g \to 0$. With the substitution (2.4.21), the local charges of the gauge theory Ansatz by Beisert, Dippel and Staudacher, and the semi-classical string theory therefore coincide to all orders.

2.4.2 The S-matrix and the Asymptotic Bethe Ansatz

Having the all-loop $SU(2)$ sector Ansatz of Beisert, Dippel and Staudacher in place, the next natural problem is to extend it to a larger sector, and ultimately to the entire theory. As explained above, in [20] Beisert was able to construct (assuming BMN scaling) the asymptotic Hamiltonian of the $SU(2)$ sector up to five loops, but apart from the two-loop $SO(6)$ sector result of [30], little success had been obtained in constructing higher loop Hamiltonians in other cases. In [61], Staudacher proposed a different approach. Instead of trying to construct the full spin chain he suggested to simply extend the Bethe Ansatz, and in particular its S-matrix, to other sectors, since this will determine the spectrum even without knowing how the dilatation operator acts. This is feasible since the S-matrices of different sectors are related for algebraic reasons.

Analyzing the semi-classical string theory S-matrices of the continuum Bethe equations of the $SU(2)$ sector of [60], the $SL(2)$ sector of [62], and the fermionic $SU(1|1)$ sector, derived from known results on near pp-wave string theory, Staudacher notes the relation

$$S_{SL(2)} = S_{SU(1|1)} S^{-1}_{SU(2)} S_{SU(1|1)} \ . \tag{2.4.27}$$

This provided a Bethe Ansatz for the $SL(2)$ sector, the topic of section 2.5, at weak coupling, which was consistent with the anomalous dimensions extracted in [63] from the QCD calculation of [64], by conjecturing that the $\mathcal{N} = 4$ dimensions coincide with the so-called maximal transcendentality piece of the QCD dimensions. Shortly thereafter, a two-loop calculation of the scaling dimension of the length three, three magnon operator [65] confirmed the prediction given by Staudacher for this dimension.

This program was then continued in [66], where an all-loop Bethe Ansatz for the entire $PSU(2,2|4)$ algebra was proposed, written in terms of the $x^{\pm}$ variables. Before introducing the Ansatz for the full algebra, the $SU(2)$, $SU(1|1)$ and $SL(2)$ sectors where unified into a $SU(1,1|2)$ sector, which is the largest subsector which does not contain interactions that change the length of the spin chain to all orders in perturbation theory. Leaving this sector one is forced to introduce dynamic effects. The structure of the S-matrix for this sector turns out to have a simple matrix form, with a global scalar factor of

$$\tilde{S}^0(x_k,\, x_j) = \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \ , \tag{2.4.28}$$

This factor is seen directly within the three closed subsectors, since their Bethe Ansatz can be written in a compact form as

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{K} \left(\frac{x_k^+ - x_j^-}{x_k^- - x_j^+}\right)^\eta \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \ , \tag{2.4.29}$$

where η takes the values 1, -1 or 0, corresponding to $SU(2)$, $SL(2)$ and $SU(1|1)$, respectively. Using the relationship

$$u_k - u_j \pm i = (x_k^\pm - x_j^\mp)(1 - g^2/x_k^\pm x_j^\mp) \tag{2.4.30}$$

one can easily recover the original $SU(2)$ sector S-matrix in terms of rapidities when $\eta = 1$.

The Asymptotic Bethe Ansatz (ABA) for the full algebra is then conjectured, by requiring it to satisfy a large number of constraints. For example, at one loop it must reduce to the one-loop Ansatz of [32], and it must give the correct result in the $SU(1,1|2)$ sector. Furthermore, a consequence of integrability, and indeed the applicability of a Bethe Ansatz, factorized scattering, mentioned above in connection with the one-loop $SU(2)$ spin chain where it was exemplified by the factorization of the three particle S-matrix in equation (2.3.20). However, when the two-particle S-matrix has a non-trivial matrix structure, including non-diagonal terms potentially changing the particle species upon scattering, it is no longer clear in which order to write the two-particle S-matrices when performing such a factorization. Consistency of integrability requires that the order not matter, producing the famous Yang-Baxter equation

$$S(p_1,\, p_2)S(p_1,\, p_3)S(p_2,\, p_3) = S(p_2,\, p_3)S(p_1,\, p_3)S(p_1,\, p_2) \ , \tag{2.4.31}$$

shown pictorially in figure 2.1, which is also satisfied by the Asymptotic Bethe Ansatz. Just as in the one-loop case the resulting Bethe Ansatz is only well defined in the zero-momentum sector, since the cyclicity constraint is necessary for the equivalence of configurations of Bethe roots corresponding to different lengths.

2.4.3 The algebraic derivation of the S-matrix

A derivation of the ABA was provided by Beisert in [58], and then more rigorously in [67], mainly by algebraic arguments. The S-matrix is constructed in an excitation picture,

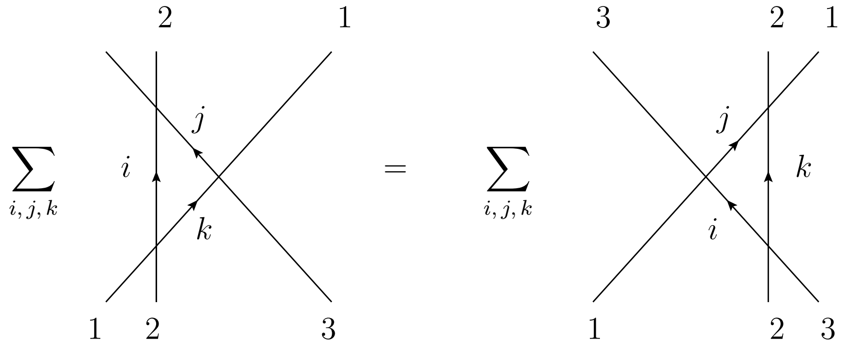

Figure 2.1: The Yang-Baxter equation. The sums are taken over all possible intermediate particles.

where one has chosen a definite vacuum state, and is only invariant under the residual algebra $psu(2|2)^2 \ltimes \mathbb{R}$, where the two $psu(2|2)$ factors share a common central charge $\mathfrak{C}$, identified with the Hamiltonian. This is in fact the largest subalgebra of $psu(2,2|4)$ which commutes with the dilatation operator[20], so if we wish to construct a spin chain with this operator as the Hamiltonian it is the largest amount of manifest symmetry we can ask for. Furthermore, the fact that the symmetry has two factors implies that one can decompose the representations as representations of each factor, and study them individually. One can therefore consider a reduced set of $(2|2)$ excitations with an S-matrix having $su(2|2)$ symmetry.

The spin chain that is considered admits the fields $\{Z,\, \phi^1,\, \phi^2|\psi^1,\, \psi^2\}$, and has an $su(2|3)$ symmetry. When a vacuum state $|0\rangle \equiv |ZZ\cdots Z\cdots\rangle$ is chosen, the symmetry that remains for the excitations $\{\phi^1,\, \phi^2|\psi^1,\, \psi^2\}$ is $su(2|2)$. The generators of the two $su(2)$ subalgebras $\mathfrak{R}^a{}_b$ and $\mathfrak{L}^a{}_\beta$ rotate the bosons and fermions among themselves in the fundamental representation,

$$
\begin{aligned}
\mathfrak{R}^a{}_b|\phi^c\rangle &= \delta^c_b|\phi^a\rangle - \frac{1}{2}\delta^a_b|\phi^c\rangle \ , \\
\mathfrak{L}^a{}_\beta|\psi^\gamma\rangle &= \delta^\gamma_\beta|\psi^\alpha\rangle - \frac{1}{2}\delta^\alpha_\beta|\psi^\gamma\rangle \ ,
\end{aligned}
\tag{2.4.32}
$$

while the supercharges $\mathfrak{Q}^\alpha{}_a$ and $\mathfrak{S}^a{}_\alpha$ rotate, as usual, fermions into bosons and vice versa.

A first problem with this approach, which will be seen in more detail below, is that the original central charge $\mathfrak{C}$ is required to take the values $C = \pm\frac{1}{2}$ in the representation $(2|2)$, and can therefore not be identified with the dilatation operator. To circumvent this problem Beisert enlarges the algebra by two additional central charges $\mathfrak{P}$ and $\mathfrak{K}$, requiring them to annihilate all physical states (implying that the symmetry of the spectrum is not modified).

In order to take into consideration the dynamic nature of the original symmetry (where states of different length where related by symmetry), markers $\mathcal{Z}^{\pm}$ are introduced into the excitation picture. These markers enter into the representation of the supercharges, and correspond to insertions or removals of Z fields in the original state. The action of the supercharges is then

$$\mathfrak{Q}^{\alpha}{}_{a}|\phi^{b}\rangle = a\delta_{a}^{b}|\psi^{\alpha}\rangle \tag{2.4.33}$$

$$\mathfrak{Q}^{\alpha}{}_{a}|\psi^{\beta}\rangle = b\varepsilon^{\alpha\beta}\varepsilon_{ab}|\phi^{b}\mathcal{Z}^{+}\rangle \tag{2.4.34}$$

$$\mathfrak{S}^{a}{}_{\alpha}|\phi^{b}\rangle = c\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi^{\beta}\mathcal{Z}^{-}\rangle \tag{2.4.35}$$

$$\mathfrak{S}^{a}{}_{\alpha}|\phi^{\beta}\rangle = d\delta_{\alpha}^{\beta}|\phi^{a}\rangle \ , \tag{2.4.36}$$

for some set of constants a, ..., d, labeling the representation. In order for the algebra to close, the condition

$$ad - bc = 1 \tag{2.4.37}$$

must be satisfied.

The new central charges $\mathfrak{P}$ and $\mathfrak{K}$ appear when commuting the supercharges and will also introduce markers when acting on the excitations. For any excitation $\mathcal{X}$

$$\mathfrak{P}|\mathcal{X}_{k}\rangle = a_{k}b_{k}|\mathcal{X}_{k}\mathcal{Z}^{+}\rangle \tag{2.4.38}$$

$$\mathfrak{K}|\mathcal{X}_{k}\rangle = c_{k}d_{k}|\mathcal{X}_{k}\mathcal{Z}^{-}\rangle \ . \tag{2.4.39}$$

Furthermore, the Hamiltonian $\mathfrak{C}$ acting on single excitation is given by

$$\mathfrak{C}|\mathcal{X}_{k}\rangle = \frac{1}{2}(a_{k}d_{k} + b_{k}c_{k})|\mathcal{X}_{k}\rangle \ , \tag{2.4.40}$$

which, by (2.4.37)-(2.4.39), would equal $\pm\frac{1}{2}$ if the additional central charges were absent.

Now, the prescription for the markers is that in the excitation picture spin chain, they should all be moved to the far right of the chain. This will introduce factors of the momenta of the excitations according to

$$|\cdots\mathcal{Z}^{\pm}\mathcal{X}_{k}\cdots\rangle = e^{\mp p_{k}}|\cdots\mathcal{X}_{k}\mathcal{Z}^{\pm}\cdots\rangle \ , \tag{2.4.41}$$

since the introduction of the Z field in the original chain will shift the excitations to the left or right, which can be undone by multiplying with such momentum factors. Using this prescription one knows how $\mathfrak{P}$ and $\mathfrak{K}$ act on the spin chain, in terms of the a_k, ..., d_k. As mentioned above, they should annihilate any physical state. It seems that the only natural way to achieve this is to set

$$a_{k}b_{k} = \alpha\left(e^{-ip_{k}} - 1\right) \ , \quad c_{k}d_{k} = \beta\left(e^{ip_{k}} - 1\right) \ , \tag{2.4.42}$$

for some constants α and β, since it then turns out that, after moving the introduced markers to the far right past all the excitations, picking up factors of the momenta along the way, the annihilation of the central charges becomes equivalent to the zero-momentum condition, satisfied by all physical states.

Solving (2.4.37), (2.4.40) and (2.4.42) for the energy gives it as

$$C = \sum_{k=1}^{K} C_k \ , \quad C_k = \pm\frac{1}{2}\sqrt{1 + 16\alpha\beta \sin^2\frac{p_k}{2}} \ . \tag{2.4.43}$$

(Another way of obtaining this expression is by writing the closure (2.4.37) as the relation $\mathfrak{C}^2 - \mathfrak{P}\mathfrak{K} = 1/4$ between the central charges, as is done in [67].) Since this should be related to the dispersion relation of $\mathcal{N} = 4$, Beisert identifies the constant $\alpha\beta$ as g^2 . However, there is actually no reason that this constant can not be a more general function of the gauge theory coupling. Still, it is very interesting that the dispersion relation emerges from this algebraic calculation. Beisert relates this to the Hamiltonian being a non-trivial part of the algebra, as discussed in [68].

Next, the a, b, c and d are parameterized in terms of the $x^{\pm}$ variables in order to be able to make contact with the known expressions in the literature. The closure (2.4.37) of the algebra then takes the form

$$x_k^+ + \frac{g^2}{x_k^+} - x_k^- - \frac{g^2}{x_k^-} = i \ , \tag{2.4.44}$$

reproducing (2.4.17), while the energy becomes and the magnon energy becomes

$$C_k = \frac{1}{2} + \frac{ig^2}{x_k^+} - \frac{ig^2}{x_k^-} = -ix_k^+ + ix_k^- - \frac{1}{2} \ , \tag{2.4.45}$$

which, up to a constant shift and constant factors, coincides with (2.4.20). The excitation S-matrix is then constructed in terms of the $x^{\pm}$ variables by requiring invariance under the algebra

$$[\mathfrak{J}_1 + \mathfrak{J}_2, \, S_{12}] = 0 \ , \tag{2.4.46}$$

where by the notation $\mathfrak{J}_i$ it is meant generator $\mathfrak{J}$ acting on site i. It should be recalled, though that $\mathfrak{J}$ may introduce a marker $\mathcal{Z}$ having a non-local effect (in these conventions picking up factors of the momenta of all particles to the right). The way that (2.4.46) is solved in practice is by acting with (2.4.46) on some two-excitation state. For example, acting on the two-scalar state $|\phi_1^a \phi_1^b\rangle$ with the $su(2)$ generators $\mathfrak{R}^a{}_b$ and $\mathfrak{L}^a{}_\beta$, and using (2.4.32), gives the structure

$$S_{12}|\phi_1^a \phi_1^b\rangle = A_{12}|\phi_2^{\{a}\phi_1^{b\}}\rangle + B_{12}|\phi_2^{[a}\phi_1^{b]}\rangle + \frac{1}{2}\varepsilon^{ab}\varepsilon_{\alpha\beta}|\psi_2^\alpha \psi_1^\beta \mathcal{Z}^-\rangle \ , \tag{2.4.47}$$

for some constants A_{12}, B_{12} and C_{12} (The insertion of the marker $\mathcal{Z}^-$ in the last term actually follows from the action of the supercharges). The rest of the algebra then determines the values of these constants, up to a global scalar factor S^0_{12}. For example, one finds

$$A_{12} = S^0_{12} \frac{x_2^+ - x_1^-}{x_2^- - x_1^+} \ , \tag{2.4.48}$$

which coincides with the $SU(2)$ sector S-matrix, as seen by setting $\eta = 1$ in (2.4.29), and choosing $S^0_{12} = \tilde{S}^0_{12}$.

The algebra thus fixes the S-matrix completely, up to a global constant. An important property of this S-matrix is that it satisfies the Yang-Baxter equation (2.4.31), as can easily be checked. Actually, in [67], the Yang-Baxter equation was used as an input to provide a more rigorous motivation of (2.4.42). It should be noted, however, that this equation usually assumes a local S-matrix which only depends on the momenta of the particles entering the scattering, while in this case other particle momenta enter as well, due to the insertions of the markers. For example, the term $S(p_1, p_2)$ on the left hand side of (2.4.31) may introduce a marker picking up a $e^{\pm i p_3}$ when commuted past the third particle. One can therefore consider the equation satisfied by the gauge theory S-matrix a twisted form of the Yang-Baxter equation [69].

The Bethe Ansatz for this spin chain can then be constructed by applying the nested Bethe Ansatz technique, in which successive “vacuums” are constructed by introducing iteratively the different types of excitations.

The string theory side and Hopf algebras.

This algebraic construction for the gauge theory rested on the existence of a discrete integrable model. Shortly thereafter, similar algebraic structures started to appear on the string side of the correspondence, without relying on a such a spin chain picture. Parts of the S-matrix had already appeared in the continuum Bethe equations in [60], for the $SU(2)$ sector, and [62] for the $SL(2)$ sector, and were further extended in for example [70] and [71], but its relation to the algebraic gauge theory determination was initiated in [72]. In this article, the properties of the string sigma model in the light-cone gauge (for which the gauge-fixed Lagrangian and Hamiltonian had been given in [70]) at infinite light-cone momentum P^+ are studied when the closed string level matching condition $p_{WS} = 0$ is relaxed. The level matching condition corresponds in the gauge theory to the zero-momentum condition of the spin chain, and in the light-cone gauge the momentum P^+ gives the circumference of the world-sheet cylinder, wherefore this setup corresponds to the scattering of asymptotic magnon states studied by Beisert in [58].

In this limit it is found that the maximal subalgebra $su(2|2) \otimes su(2|2)$ of $psu(2,2|4)$

which commutes with the Hamiltonian receives a central extension c when going off-shell (relaxing the level matching condition). The central extension was calculated and found to be

$$c = \frac{1}{2\zeta}(e^{ipws} - 1) \;, \tag{2.4.49}$$

where $\zeta = \frac{2\pi}{\sqrt{\lambda}}$, in accordance with the centrally extended algebra of the gauge theory construction (where the other conserved charge is identified with the complex conjugate of c).

One aspect of the algebra which is not present for infinite P^+, however, is its length changing effects. Such effects are related to the zero-mode of the non-physical light-cone field $x_-(s)$ when P^+ is finite, and should related to the fact that the cyclicity constraint is necessary for the consistency of the gauge theory Bethe ansatz when dynamical effects are included. This does not seem consistent with the gauge theory S-matrix, since it is dynamic, introducing, as in equation (2.4.47), $\mathcal{Z}$-markers.

This difference was then made more concrete in [73], where the scattering matrix was calculated directly for excitations in the light-cone gauge at infinite P^+ at first order at strong coupling, a calculation which was then extended to two loops in [74] in the so-called near-flat [6] limit [75]. The resulting S-matrix had a centrally extended $psu(2|2) \otimes psu(2|2)$ symmetry, but the S-matrix was not dynamical and satisfied a standard (as apposed to twisted) form of the Yang-Baxter equation. The representation of the symmetry on multi-particle states is consistent with a Hopf algebra structure. Among other things, this implies the existence of a non-trivial co-product $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, which takes an element of the algebra into the tensor product of the algebra with itself, and thus provides a non-trivial action of the algebra on tensor product representations. Representations on triple tensor products, and analogously for multiple tensor products in general, are defined by $(1 \otimes \Delta)\Delta$, which is unambiguous due to the property

$$(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta \tag{2.4.50}$$

of Hopf algebras, known as co-associativity.

In fact, before the appearance of these string theory results, the papers [76] and [77] discussed the Hopf algebra structure implied by the introduction of the markers in the gauge theory setup. In doing so one introduces a new generator $\mathfrak{U}$, which acts on any excitation $\mathcal{X}_k$ by returning e^{ip_k}. The action of the central charges (2.4.38) and (2.4.39) on multiple excitations can then be described in the co-product notation as

$$\Delta\mathfrak{P} = \mathfrak{P} \otimes \mathfrak{U}^{-1} \;, \quad \Delta\mathfrak{K} = \mathfrak{K} \otimes \mathfrak{U} \;, \tag{2.4.51}$$

[6] In this limit $\lambda \to \infty$ with the magnon momenta scaling as $p \sim \lambda^{-1/4}$.

while

$$\Delta \mathfrak{U} = \mathfrak{U} \otimes \mathfrak{U} \ . \tag{2.4.52}$$

In [76] this analysis was limited to the action of the central charges, while in [77] it was applied to the $su(1|2)$ subalgebra of $su(2|2)$. It is not possible, however, to define a co-product which directly reproduces the (2|2) representation, since in (2.4.33)-(2.4.36) the introduction of the markers depends on which state is acted on.

In [67], however, the more rigorous derivation of the $su(2|2)$ invariant S-matrix included an additional set of markers $\mathcal{Y}^{\pm}$, which are not forbidden by purely algebraic arguments. The action of the supercharges (2.4.33)-(2.4.36) is then generalized to

$$\mathfrak{Q}^{\alpha}{}_{a,\,k}|\cdots\phi^{b}_{k}\cdots\rangle = a\delta^{b}_{a}|\cdots\psi^{\alpha}_{k}\mathcal{Y}^{+}\cdots\rangle \tag{2.4.53}$$

$$\mathfrak{Q}^{\alpha}{}_{a,\,k}|\cdots\psi^{\beta}_{k}\cdots\rangle = b\varepsilon^{\alpha\beta}\varepsilon_{ab}|\cdots\phi^{b}_{k}\mathcal{Z}^{+}\mathcal{Y}^{-}\cdots\rangle \tag{2.4.54}$$

$$\mathfrak{S}^{a}{}_{\alpha,\,k}|\cdots\phi^{b}_{k}\cdots\rangle = c\varepsilon^{ab}\varepsilon_{\alpha\beta}|\cdots\psi^{\beta}_{k}\mathcal{Z}^{-}\mathcal{Y}^{+}\cdots\rangle \tag{2.4.55}$$

$$\mathfrak{S}^{a}{}_{\alpha,\,k}|\cdots\phi^{\beta}_{k}\cdots\rangle = d\delta^{\beta}_{\alpha}|\cdots\phi^{a}_{k}\mathcal{Y}^{-}\cdots\rangle \ . \tag{2.4.56}$$

Choosing $\mathcal{Y}^{\pm} = 1$ is natural from the gauge theory, while the light-cone worldsheet string theory S-matrix is recovered when one chooses $\mathcal{Y} = \sqrt{\mathcal{Z}}$. And this is precisely what is needed in order for the action (2.4.53)- (2.4.56) to be compatible with a Hopf algebra[78], since, for a given generator, the introduction of the markers then becomes independent of the field that is acted upon. Furthermore, this choice of $\mathcal{Y}$ marker removes the insertions of markers by the S-matrix, and the twisted Yang-Baxter equation becomes an ordinary Yang-Baxter equation.

The algebraic construction of the string theory S-matrix was performed rigorously in [69], where the difference between the gauge and string theory was elucidated, and a non-local change of basis between the two was given. The basis corresponding to each theory was called the gauge frame and string frame, respectively, and since they were related by a change of basis, the resulting spectra are equivalent.

In conclusion, imposing integrability, the algebra determines an all-loop Bethe Ansatz to a large extent, which seems to coincide between the string and gauge theories. Some additional input must still be given, however. To start with, the rapidity map, relating the rapidity u of the one-loop $SU(2)$ S-matrix to the $x^{\pm}$ variables must be provided. A comparison between string and gauge theories would suggest that (2.4.15) is the correct choice, but this must still be proved. Also, this algebraic construction does not determine the overall scalar factor S^{0}_{12}, which could in principle differ from gauge and string theories. Indeed, by the time the algebraic construction was performed, it was no longer believed that the overall scalar factor was equal to (2.4.28). Instead, it was found necessary to add a non-trivial "dressing factor", the topic of section 2.5.3, in order to reconcile string and gauge theory results.

2.5 The $SL(2)$ sector and the single magnon at unit spin.

In this section we will dedicate some attention to the $SL(2)$ sector, as it is very important in the present context. Also, towards the end of the section, we will present some of our own results regarding a relationship between the single-magnon energy and the spectrum of anomalous dimensions in this sector.

As mentioned above, in [31], which was the first time that the $SL(2)$ sector appeared under this name in the context of $\mathcal{N} = 4$, Beisert lifted the known $SL(2)$ Hamiltonian to the entire theory in constructing the complete one-loop algebra. Furthermore, much of the evidence for the Asymptotic Bethe Ansatz, together with successful interpolations between gauge and string theory results, have been produced in this sector. Also, as will be explained below, the sector provides a connection between anomalous dimensions and the BFKL framework for the high energy limit of scattering amplitudes. We will also present a conjecture that the energy for the single magnon at momentum $p = \pi$ can be obtained directly from the physical spectrum of this sector.

In the gauge theory, the $SL(2)$ sector contains operators which are a sum of terms of the form

$$\mathrm{Tr}\mathcal{D}^{m_1} Z \mathcal{D}^{m_2} Z \cdots \mathcal{D}^{m_L} Z \ , \tag{2.5.1}$$

where $m_1 + \cdots + m_L = M$, and

$$\mathcal{D} \equiv \mathcal{D}_{1+i2} = \mathcal{D}_1 + i\mathcal{D}_2 \tag{2.5.2}$$

is a complex combination of covariant derivatives. Here, the operator length L is referred to as the twist, which is defined as the classical dimension minus the spin, which gives $L + M - M = L$ for these operators. Twist is an important concept as, for example, in an operator product analysis of deep inelastic scattering operators of low twist will dominate. The sector is closed under renormalization because it saturates the bound $\Delta_0 \geq L + M$, where M is the charge with respect to rotations in the spacetime 12-plane, and L is the R-charge corresponding to rotations in the flavor 56-plane, and operators of different charges under the global symmetries do not mix.

In terms of the generators of the $psu(2,2|4)$ algebra, the $sl(2)$-algebra is generated by

$$J'_+ = P_{11} = P_{1+i2} \tag{2.5.3}$$

$$J'_- = K^{11} = K^{1+i2} \tag{2.5.4}$$

$$J'_3 = \frac{1}{2}D + \frac{1}{2}\delta D + \frac{1}{2}L^1{}_1 + \frac{1}{2}L^2{}_2 = D + \frac{1}{2}\delta D - \frac{1}{2}L \ . \tag{2.5.5}$$

In terms of creation and annihilation operators, the algebra can be represented as

$$J'_+ = \boldsymbol{a}^\dagger + \boldsymbol{a}^\dagger\boldsymbol{a}^\dagger\boldsymbol{a} \ , \quad J'_- = \boldsymbol{a} \ , \quad J'_3 = \frac{1}{2} + \boldsymbol{a}^\dagger\boldsymbol{a} \ . \tag{2.5.6}$$

The states

$$(\boldsymbol{a}^\dagger)^n|0\rangle \equiv \frac{1}{n!}\,(\mathcal{D}_{1+i2})^n\, Z \tag{2.5.7}$$

then form an infinite dimensional spin $j = -1/2$ representation of this algebra, with highest weight $2j = -1$.

At one-loop the $SL(2)$ spin chain coincides with the Heisenberg XXX chain at spin $-1/2$ which had been studied earlier, in for example [79], due to the appearance of integrable $SL(2)$ chains in QCD [41], while the one-loop Hamiltonian, given in equations (2.3.30) and (2.3.31), appeared first in [40]. Its Bethe Ansatz was then extended to higher loops for $\mathcal{N} = 4$ in [61] and [66], while the two-loop Hamiltonian was given in [80]. The all-loop Bethe Ansatz for the sector is given by (2.4.29) with $\eta = -1$, and M and L defined as in (2.5.1), but, in contrast with the $SU(2)$ sector, M can now be larger than L.

Perhaps the most interesting, and surprising, property of the $SL(2)$ sector is that, at least for the first few L, exact expressions for the spectrum (of the lowest lying states) can be given at the first perturbative orders, valid for any M. Let us define $\gamma^{(L)}(M)$ as the lowest anomalous dimension corresponding to M and L, write its perturbative expansion as

$$\gamma^{(L)}(M) = \sum_{r=1}^{\infty} \gamma_r^{(L)}(M) g^{2r} \ . \tag{2.5.8}$$

In the case of twist-two operators, one then has for the first three loops [42, 63]

$$\begin{aligned}
\gamma_1^{(2)}(M) \ &= \ 8\,S_1 \ , && (2.5.9)\\
\gamma_2^{(2)}(M) \ &= \ -16\left(S_3 + S_{-3} - 2S_{-2,1} + 2S_1(S_2 + S_{-2})\right) \ , && (2.5.10)\\
\gamma_3^{(2)}(M) \ &= \ -64\left(2S_{-3}S_2 - S_5 - 2S_{-2}S_3 - 3S_{-5} + 24S_{-2,1,1,1} + 6(S_{-4,1} + S_{-3,2} + S_{-2,3})\right.\\
&\quad -S_1(8S_{-4} + S_{-2}^2 + 4S_2S_{-2} + 2S_2^2 + 3S_4 - 12S_{-3,1} - 10S_{-2,2} + 16S_{-2,1,1})\\
&\quad \left.-12(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}) - (S_2 + 2S_1^2)(3S_{-3} + S_3 - 2S_{-2,1})\right) \ , && (2.5.11)
\end{aligned}$$

where the symbols S are harmonic sums, defined through

$$S_a(M) \equiv \sum_{j=1}^{M} \frac{(\mathrm{sgn}(a))^j}{j^{|a|}} \ , \quad S_{a_1,\ldots,a_n}(M) \equiv \sum_{j=1}^{M} \frac{(\mathrm{sgn}(a_1))^j}{j^{|a_1|}} S_{a_2,\ldots,a_n}(j) \ . \tag{2.5.12}$$

An interesting, and to date not properly understood, property of (2.5.9)-(2.5.11) is that the sum of the absolute values of the indices of the harmonic sums in any term is always $2r-1$. This sum is called the "transcendentality", and stems from its association to numbers such as the values of the Riemann zeta function $\zeta(a)$. The number $\zeta(a)$ is defined to have transcendentality a, which is consistent with the definition of the transcendentality of the harmonic sums since

$$\lim_{M\to\infty} S_a(M) = \zeta(a) \ , \tag{2.5.13}$$

a definition which turns out to be crucial since, as we will see later, starting from four loops values of the zeta function appear in the coefficients multiplying the harmonic sums. Transcendentality simplifies much of the analysis of the $SL(2)$ sector, and in fact, the authors of [42] and [63] used this hypothesis when extracting (2.5.9)-(2.5.11) by keeping the maximal transcendentality piece of a much more involved QCD computation [64]. Several years later, the three-loop spectrum was derived from the higher-order Baxter equation (equivalent to the Asymptotic Bethe Anzatz) without relying on the transcendentality assumption in [81].

Also, similar expressions have been calculated for twist-three operators [82], which obey the same transcendentality hypothesis. There is some additional structure in this case, however, since it turns out that the twist-three spectrum can be written in a basis of harmonic sums of only positive indices, evaluated at $\frac{M}{2}$. The one-loop spectrum is, for example, given by

$$\gamma_1^{(3)}(M) = 8 S_1\left(\frac{M}{2}\right) \ . \tag{2.5.14}$$

Now, an important reason for studying the $SL(2)$ sector in the context of AdS/CFT is that it can provide new checks for the correspondence. Having the all-loop ABA one would like to be able to extract anomalous dimensions at strong coupling in order to compare with string results. A problem with this approach is that the ABA is precisely asymptotic, meaning that it is not valid beyond a certain perturbative order in which wrapping effects, the topic of section 2.6, enter. However, in [83] it was argued that the high-spin limit (large M limit in our notation) of twist operators should be independent of L. This means that one should be able to use the ABA to calculate the cusp anomalous dimension, given, as shown in [6], by the high-spin limit of twist-two operators, to all orders since the L-universality allows one to choose L sufficiently large at each perturbative order as to avoid wrapping effects. In [84] it was argued in general that the high-spin limit of the anomalous should grow like a logarithm of the spin, and the cusp anomalous dimension then appears as the coefficient multiplying that logarithm.

The first attempt to calculate the all-loop cusp anomalous dimension was given in [85]. In taking the large spin limit, the discrete Bethe equations become an integral equation for a continuos root density. Special care had to be taken in treating the one-loop excitation density, since the integral equation becomes singular in this limit, a phenomenon which had already been studied in [86] when treating the spin 0 $SL(2)$ chain describing scattering of reggeized gluons, the topic of section 5.2.

The influence of the higher order corrections to the Bethe equations can then be recast into an integral equation (the ES equation) for the fluctuations $\hat{\sigma}(t)$ of the excitation density around the one-loop solution. The cusp anomalous dimension is then given as

$$f(g) = 8g^2 - 16g^4 \int_0^\infty dt\, \hat{\sigma}(t) \frac{J_1(2gt)}{2gt} \ , \tag{2.5.15}$$

where $J_1(x)$ is the Bessel function of the first kind. One can easily obtain a weak coupling solution to the ES equation by iteration, with the result that

$$f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 - 16\left(\frac{73}{630}\pi^6 - 4\zeta(3)^2 + 8\beta\zeta(3)\right) g^8 + \dots \ , \tag{2.5.16}$$

which coincides, up to three loops, with the amplitude based calculation [7]. The four-loop term contains an unknown constant β, corresponding to the freedom one has in the ABA of choosing the overall scalar factor. If BMN scaling is to be preserved, as was the assumption behind the all-loop Bethe Ansatz of Beisert, Dippel and Staudacher, β is required to be zero. As will be explained in section 2.5.3, this turns out not to be the case, and the ES equation is corrected by a non-trivial dressing factor[87].

On the string side, the first article treating the $SL(2)$ sector was [88]. As for the $SU(2)$ sector, the R-charge L is interpreted as an angular momentum on the five sphere. The novelty is that one now has a spin, given by M, in AdS_5. For states of large enough M, the energy should, as for the gauge theory, be independent of L. One can therefore consider strings spinning in AdS_5 with no motion on S^5. Furthermore, the cusp anomalous dimension is the smallest anomalous dimension for high spins, so the string states of relevance are closed strings on the leading Regge trajectory, i.e. those states that have the smallest mass for a given spin. In flat space it is known that such states are given by folded strings spinning as a rigid rod around their centers. Here, the flat space is replaced by the AdS-space and in global coordinates,

$$ds^2 = R^2 \left[-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 \right] \ , \tag{2.5.17}$$

and one considers a spinning folded closed string whose center lies at rest at $\rho = 0$. For

$M >> \sqrt{\lambda}$ one finds

$$\Delta - M = \frac{\sqrt{\lambda}}{\pi} \ln\left(\frac{M}{\sqrt{\lambda}}\right) + \mathcal{O}(M^0) \ , \tag{2.5.18}$$

which is interesting since the logarithmic growth of the anomalous dimensions associated with gauge theories appears. The coefficient of the logarithmic term of (2.5.18) thus gives the leading strong coupling term of the cusp anomalous dimension. One, and two-loop calculations of the energies of the string spinning with a large spin in AdS are performed in [90] and [91], respectively, with the result that the energy continues to grow proportionally to the logarithm of the spin, and with the cusp anomalous dimension given by

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\ln 2}{\pi} - \frac{K}{\pi}\frac{1}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda}\right) \ , \tag{2.5.19}$$

where K is the Catalan constant. A simple explanation for the logarithmic growth with the spin was later given in [89], based on the geometry of the AdS-space.

2.5.1 Analytical continuations of the $SL(2)$ spectrum and BFKL

Expressions such as (2.5.9)-(2.5.11) are defined for positive, integer M, and it would not seem to make sense to consider other values. However, a connection to high energy scattering motivates an extension of their definition to more general M. It turns out that [11] for (and only for) $\mathcal{N}=4$ the BFKL framework [10] for high energy scattering, which we introduce in chapter 5, is related by analytical continuation to the DGLAP equation [93], providing the evolution of parton distribution functions with the energy scale. Without going into details, a matrix of anomalous dimensions of twist-two operators is obtained as a Mellin transform

$$\gamma_{ab}(M) = \int_0^1 dz\, z^{M-1} P_{ab}(z) \tag{2.5.20}$$

of the splitting kernels $P_{ab}(z)$ entering into the DGLAP equation. This provides a relationship between the anomalous dimensions of the $SL(2)$ sector and BFKL. Actually, BFKL provides an object known as the "BFKL anomalous dimension" γ_{BFKL}, which can be interpreted as an anomalous dimension of an operator having the same quantum numbers as an operator of the type (2.5.1), but at spin $M=-1$! The claim is that it is possible to recover this object by analytically continuing the $SL(2)$ spectrum to negative M.

In [92] the way to analytically continue harmonic sums to arbitrary values of M was discussed. It turns out that there is a well-defined way to analytically continue sums with positive indices. Sums with a negative index, however, such as $S_{-a,b,\ldots}$, have, due to their definition as an alternating series, a $(-1)^M$ factor. The oscillatory nature of this factor would, after analytical continuation, make the sums explode exponentially along the imaginary M axis, which does not allow for a physical interpretation. The reason is that the inverse Mellin transform, which should produce the splitting functions of the DGLAP equation, is taken along the imaginary axis and would be divergent if such an oscillatory factor where present. In order to obtain something physically sensible one can instead choose to analytically continue solely from even (or odd) values of M, setting the $(-1)^M$ factor to a constant +1 (-1).

In the notation of [92], the harmonic sums obtained by continuing from even M are denoted S^+ (together with the corresponding indices) and the sums obtained from negative M are written S^-. It should be stressed that S^+ (S^-) give incorrect values for odd (even) integer M. The two prescriptions then define two analytic expressions for the anomalous dimensions, $\gamma^{(+)}(M)$ and $\gamma^{(-)}(M)$.

In QCD, both expressions are present, in the form of the singlet and non-singlet anomalous dimensions. But should the (+) or (−) prescriptions be chosen in the case of $\mathcal{N}=4$ SYM? The applicability of the equations (2.5.9)-(2.5.11) provide a possible answer since they only give the anomalous dimensions for even M. For odd M they turn out to be incorrect because it can be shown that operators with an odd number of covariant derivatives do not acquire an anomalous dimension, and should therefore have $\gamma = 0$. A non-zero spectrum is however what is obtained from the spin chain and the odd M are therefore considered un-physical. Requiring that physical states are related by analytical continuation singles out the (+) prescription.

Analytically continued to negative M, the harmonic sums diverge. For example,

$$S^+_{-a}(M+\omega) \sim \frac{(-1)^{M+1}}{\omega^a}\;, \quad \omega \to 0\;, M=-1,\,-2,\,\dots\;. \tag{2.5.21}$$

The spectrum can then be compared to γ_{BFKL}, which, applying leading and next-to-leading order BFKL, gives the leading and next-to-leading singular behavior at $M=-1+\omega$ to all orders in perturbation theory! To the lowest four orders, one has

$$\gamma_{BFKL} = (-8+0\omega)\frac{g^2}{\omega}+(0+0\omega)\left(\frac{g^2}{\omega}\right)^2-(0+\zeta(3)\omega)\left(\frac{4g^2}{\omega}\right)^3-(4\zeta(3)+5/4\zeta(4)\omega)\left(\frac{4g^2}{\omega}\right)^4+\cdots\;. \tag{2.5.22}$$

Up to three loops this is precisely what is obtained by continuing (2.5.9)-(2.5.11) to $M=-1+\omega$ using the (+) prescription.

Given that it seems that the spectrum of anomalous dimensions in the $SL(2)$ sector are consistent with the predictions of BFKL up to three loops it is reasonable to expect this relationship to hold to higher loops as well. The equation (2.5.22) therefore provides a constraint that the four-loop anomalous dimension should satisfy. This was famously used in [12] to show for the first time that the Asymptotic Bethe Ansatz was indeed limited by wrapping effects starting from four loops by comparing the analytical continuation of its prediction for the four-loop spectrum with (2.5.22). The full four-loop ABA expression for the twist-two spectrum is long, and can be found in [12], but its leading singular behavior comes from the terms

$$256\,(4S_{-7}(M)+6S_7(M))\ , \tag{2.5.23}$$

which from (2.5.21) are seen to give $\frac{-512}{\omega^7}$ at $M=-1$. This violates (2.5.22) in the maximal possible way, given a behavior that is too singular, as singular as transcendentality permits. In the more recent articles [94] and [13], proposals for the wrapping correction, discussed in section 2.6, to the ABA spectrum are given at four and five loops, respectively, in perfect agreement with the constraints from BFKL. It should also be noted that in the latter case, BFKL is practically the only check that exists for the proposed spectrum, due to the difficulty of performing explicit five-loop calculations.

2.5.2 The appearance of the single magnon

The analytical continuation which seems natural from the viewpoint of BFKL is the (+) prescription that links physical states of the $SL(2)$ spin chain. However, from the viewpoint of the $SL(2)$ spin chain, we can also view odd M as physical, and the odd analytical continuation links such states. An alternative way of understanding why these odd M are unphysical in the gauge theory is that they will correspond to states of non-zero total momentum, and therefore do not satisfy the cyclicity constraint which, as explained before, is necessary in order for the entire gauge theory to be consistent, including dynamical effects. Later in this section we will explain, for example, why the total magnon momentum is always π for odd M in the case of twist two.

The reason for considering these odd M states is an observation made by us in [95] that if one evaluates the harmonic sums in (2.5.9)-(2.5.11) at $M=1$ (an evaluation that is trivial since $S_{a_1,\ldots,a_n}(1)=\mathrm{sgn}(a_1)\cdots\mathrm{sgn}(a_n)$), one obtains

$$\gamma^{(2)}(1)=8g^2-32g^4+256g^6+\cdots\ , \tag{2.5.24}$$

which curiously coincides with the expansion of the all-loop dispersion relation (2.4.14) for a magnon of momentum $p=\pi$. Extrapolating, we can therefore conjecture that

$$E(p=\pi)=\gamma^{(2)}(1) \tag{2.5.25}$$

to all orders in perturbation theory. Of course, starting from four loops wrapping effects enter and we can then no longer expect (2.4.14) to be valid, but (2.5.25) can still constrain the spectrum, relating anomalous dimensions to single magnon energies. Such single magnon energies become physical observables when considering the β-deformed theory, the topic of section 3, but can even in the non-deformed theory be obtained from the dilatation operator. Since the $SL(2)$ sector is consistent even when the cyclicity constraint is relaxed, one can act directly with the dilatation operator on single magnon states. Perhaps the most useful application of (2.5.25) is then to constrain the dilatation operator, since the twist-two spectrum is known up to five loops [94, 13], but the dilatation operator is not known at such high orders.

Now, why does the momentum $p = \pi$ appear? An explanation can be found in the Baxter equation. The one-loop Bethe Ansatz for the $SL(2)$ sector is

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j \neq k}^{M} \frac{u_k - u_j - i}{u_k - u_j + i} \ , \tag{2.5.26}$$

where the rapidity u, as before, is related to the momentum as $u = \frac{1}{2} \cot \frac{p}{2}$ Given a solution $\{u_k\}$, $k = 1, \ldots, M$, we can introduce the Baxter Q-function

$$P_M(u) = \alpha \prod_k (u - u_k) \ , \tag{2.5.27}$$

where we use the notation P_M, following [94], instead of the usual Q_M notation, and where α is a normalization constant. The Bethe equations (2.5.26) can then be re-written

$$\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = -\frac{P_M(u_k - i)}{P_M(u_k + i)} \ , \tag{2.5.28}$$

where the minus sign on the right hand side stems from including the term , missing in (2.5.26), in which $j = k$. Rearranging, this implies that the polynomial of degree $L + M$

$$A(u) \equiv (u + i/2)^L \, P_M(u + i) + (u - i/2)^L \, P_M(u - i) \tag{2.5.29}$$

must have zeros at the Bethe roots u_k. We can therefore factor out $P_M(u)$ and write $A(u) = t(u)P_M(u)$, for some polynomial of degree L, $t(u)$, usually called the transfer matrix. Putting everything together we arrive at the Baxter equation:

$$(u + i/2)^L \, P_M(u + i) + (u - i/2)^L \, P_M(u - i) = t(u)P_M(u) \ . \tag{2.5.30}$$

We have seen how the Bethe equation imply the Baxter equation and the inverse is also true since the equation requires the equality of the coefficients of two polynomials of order $L + M$, providing precisely enough equations to determine the M Bethe roots u_k and the $L + 1$ coefficients of $t(u)$. Some of the latter are in fact determined directly. Matching the two highest orders in u will directly give the coefficient of u^L and u^{L-1} as 2 and 0, respectively, so that

$$t(u) = 2u^L + t_{L-2}u^{L-2} + t_{L-3}u^{L-3} + \cdots + t_0 \ . \tag{2.5.31}$$

Curiously, also at order $L + M - 2$ the u_k cancel out and one obtains that

$$t_{L-2} = -\frac{1}{4}L(L-1) - LM - M(M-1) \ . \tag{2.5.32}$$

Our reason for introducing the Baxter equation is that, in contrast with the Bethe equations, it makes sense for $M = 1$. For $L = 2$ and $M = 1$ the unique solution for the single root is precisely $u_1 = 0$, corresponding to momentum π, justifying why the $p = \pi$ magnon appears at $M = 1$.

In fact, at twist two and odd M the total magnon momentum is always π. Indeed, both the LHS and the RHS of (2.5.30) are odd under the simultaneous change of sign $u \to -u$, $u_k \to -u_k$. This implies that at even (odd) orders in u only an odd (even) number of factors of the u_k can appear in each term. Furthermore, at order $M - n$ one can have at most n factors of the roots, since it follows from the Baxter equation that the terms with $M-n+2$ factors of the u_k will cancel . Basically, on the LHS such terms never pick up the $\pm i$ in the arguments of the Baxter functions since this gives fewer factors of the u_k, and always contain the u^L terms from the expansion of $(u \pm i/2)^L$, leading to a cancellation with the $2u^2 P_M(u)$ of the RHS. In consequence, the equation obtained at order $M - 1$ is

$$\sum_k u_k = 0 \ . \tag{2.5.33}$$

Since the Baxter equation is also symmetric under all permutations of the u_k, the equation obtained at order $M - 3$ is then

$$a \sum_k u_k + b \sum_\sigma u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} \ , \tag{2.5.34}$$

where we sum over all permutations σ of the indices $1\,2,\ \ldots,\ M$, and where a and b are two numbers. Applying (2.5.33) we can discard the first term, and since b can not be zero, as this would imply the Bethe equations to have a continuous family of solutions, we obtain

$$\sum_\sigma u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} = 0. \tag{2.5.35}$$

Continuing this process iteratively, for all odd n we have

$$\sum_\sigma u_{\sigma(1)} \cdots u_{\sigma(n)} = 0 \ , \tag{2.5.36}$$

and, in particular, for odd M we can set $n = M$ (corresponding to the zeroth order term in the Baxter equation) which gives

$$u_1 u_2 \cdots u_M = 0 \ , \tag{2.5.37}$$

showing that at least one of the u_k must be zero. The only solution to the equations (2.5.36) is that the non-zero roots then be paired, the negative of each root also being a root. Since changing the sign of a root is equivalent to changing the sign of the corresponding momentum this implies that, except for a lone $p = \pi$ magnon all momenta cancel, and the total momenta becomes precisely π.

Given the appearance of the magnon in the twist-two spectrum one might want to look at higher twists as well. The Baxter equation can easily be solved for $M = 1$ in those cases as well. For $L = 3$ one finds solutions corresponding to $p = \pm 2\pi/3$, while twist 4 exhibits three solutions, of momenta $p = \pi$ and $p = \pm\pi/2$, respectively. Furthermore, at twist 5 we have the solutions $p = \pm\frac{2\pi}{5}$ and $p = \pm\frac{4\pi}{5}$. This trend continues in general: setting $u = u_1$ in (2.5.30) when $M = 1$ gives

$$(u_1 + i/2)^L - (u_1 - i/2)^L = 0 \ , \tag{2.5.38}$$

which for finite u_1 is equivalent to ordinary momentum quantization of the single magnon. The magnon of minimum energy, the zero-momentum one, corresponding to $u_1 = \infty$, is however not a solution to this equation since the leading behavior of the LHS at large u_1 is iLu^{L-1}. Also, all of the solutions of (2.5.38) will be solutions of the entire Baxter equation since, as long as the LHS of (2.5.30) has a zero at $u = u_1$, the polynomial $t(u)$ can be chosen appropriately.

Since $\gamma^{(L)}(M)$ is defined as the M magnon solution of minimum anomalous dimension, it is then tempting to conjecture that $\gamma^{(L)}(1)$ corresponds to those solutions having minimum energy according to the dispersion relation. This suggests, as was also conjectured in [95], that we extend (2.5.25) to arbitrary twist as

$$E\left(p = \frac{2\pi}{L}\right) = \gamma^{(L)}(1) \ . \tag{2.5.39}$$

This statement is stronger that what can be derived from asymptotic considerations since it provides an all-loop constraint, including wrapping corrections, on the higher twist spin chain spectrum, which are beyond the reach of the Baxter equation and its higher-loop generalizations.

At first sight, however, this more general conjecture would seem to be incorrect since it is not satisfied by the one-loop twist-three formula (2.5.14). The analytical continuation of S_1 is well-defined and unique with $S_1\left(\frac{1}{2}\right) = 2(1 + \ln 2)$. However, one must remember that (2.5.39) is a statement about the odd M states of the spin chain, while (2.5.14) is only valid for even M. Indeed, when setting $M = 1$ in the twist-two formula the harmonic sums where evaluated using their original definition, which at odd M corresponds to choosing the $(-)$ analytical prescription. It is very curious that for twist 2 the odd and even M spectra are given by the same formula, but this does not have to happen in general.

This suggests that perhaps one should not forget about the $\gamma^{(-)}$ prescription in $\mathcal{N} = 4$ SYM, as it might be the correct continuation for certain non-physical states. And since such non-physical states could be related, by analytic continuation, to physical states in other sectors of the theory, it may turn out to be of crucial importance. For example, in [12] it was shown that the Asymptotic Bethe Ansatz anomalous dimensions for twist two at four loops agrees, using the $(+)$-prescription for analytic continuation, with another constraint coming from the BFKL framework, the so-called double logarithms, at negative, even values of M. They are not found to agree for odd, negative values of M, though. However, if we use the $(-)$-prescription, it is easy to check that we do find agreement with the odd, negative M. One speculative interpretation of this is that the even, negative M double logarithms are related to the even, positive M through the $(+)$-prescription, while the odd, negative M double logaritms are connected, through $\gamma^{(-)}$ with unphysical states at odd, positive M. One such state is the one-magnon state underlying the conjecture (2.5.25).

2.5.3 The dressing factor

As fascinating as the higher-loop, dynamical spin chain integrability may be, as a check of the AdS/CFT correspondence it was only partially successful at the time. One the one hand, in [96] it was shown that the integrable string sigma model has a so-called Yangian symmetry[7], which could also be found at one-loop in the gauge theory, an observation that was extended to two loops in [97], and it was shown in [98] that not only do the one-loop scaling dimensions of large dimension operators agree with the energy of spinning strings on $AdS_5 \times S^5$, but the eigenvalues of an infinite set of higher conserved charges of the integrable models also coincide for these configurations. Furthermore, spectral curves classifying all states of the classical string theory [60, 99] and all local operators in the thermodynamic limit of the gauge theory [100] were found to agree.

On the other hand, in [101], using the Bethe Ansatz a three-loop gauge theory calculation was performed in the $SU(2)$ sector, with a puzzling discrepancy appearing from three loops with the corresponding string theory result of [37]. Also, in [102] corrections to the pp-wave limit were calculated on the string side, which did not coincide with the gauge theory starting from three loops, and in [103] the values of the conserved charges corresponding to the folded and circular spinning string solutions were shown to differ with the gauge theory starting from this order. These discrepancies were proposed to be due to an order of limits problem in [56], suggesting that the two limits $L \to \infty$ and $\lambda \to 0$, taken when comparing gauge and string theories, might not commute, a possibility that had already been discussed in [104], in the context of the BMN limit.

The solution to this mismatch was the introduction of a non-trivial correction to the overall BDS scalar factor (2.4.28), known as the dressing factor. The first time this factor appeared was in the article [105] by Arutyunov, Frolov and Staudacher, in which the

[7] We will define what a Yangian symmetry algebra is in section 4.4, in the context of scattering amplitudes.

integral Bethe equations of [60] were discretized, taking the gauge theory Bethe Ansatz as a guide, producing a set of discrete Bethe equations reproducing many known string theory results at the time. The S-matrix obtained in this way differed from that of the gauge theory BDS Ansatz, starting from three loops. Writing

$$\tilde{S}_{\text{string}}(p_k,\, p_j) = \tilde{S}_{\text{BDS}}(p_k,\, p_j) e^{i2\theta(p_k\, p_j)} \ , \tag{2.5.40}$$

the additional phase takes the form

$$\theta(u_1,\, u_2) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \beta_{r,\,s}(\lambda) \left(q_r(u_1) q_s(u_2) - q_s(u_1) q_r(u_2) \right) \ , \tag{2.5.41}$$

where the coefficients $\beta_{r,s}(\lambda)$ are expanded as

$$\beta_{r,s}(\lambda) = \sum_{k=s-1}^{\infty} \lambda^k \beta_{r,s}^{(k)} \ , \tag{2.5.42}$$

and the q_r are the ordinary gauge-theory conserved charges given in terms of $x^{\pm}$, as proposed in [56]. In the original proposal, the β coefficients took the simple $\beta_{r,\,s}(\lambda) \propto \lambda^r$, giving what later became known as the AFS phase. But Arutyunov, Frolov and Staudacher also suggested that the gauge and string theory results might be reconciled by an interpolating Bethe Ansatz obtained by allowing the $\beta_{r,\,s}$ to be more general functions of the coupling.

This was made more precise in [106], where an analysis of the possible integrable long-range spin chains allowed by the gauge theory was carried out, with the conclusion that the dressing phase, taken as the residual scalar phase allowed by the algebraic construction, will take the form (2.5.41). Also, in [106], it was shown that the dressing enters starting from three loops. This explains why the observed gauge/string discrepancies started from three loops in the gauge theory.

But apart from the comparisons with the gauge theory, It was not long until the necessity of a non-trivial dressing phase was noted purely from the point of view of the string theory. The AFS phase should receive a correction at order $\frac{1}{\sqrt{\lambda}}$ in order for the AFS Bethe Ansatz to correctly reproduce a set of of one-loop string theory calculations [107]. The full one-loop correction to the AFS phase was then calculated by Hernandez and Lopez in [108], producing what is called the HL phase. This phase was then re-derived rigorously in [109].

Of course, in order to connect the gauge and string theories one must have a means to determine an all-loop expression for the dressing phase. Such a means was proposed by Janik in [110], by making the observation that the dressing factor of S-matrices in relativistic integrable quantum field theories were frequently determined (up to so-called

CDD factors) by crossing symmetry. If it were possible to quantize the string world sheet theory in covariant gauge it should exhibit crossing symmetry, thereby producing constraints on the overall scalar factor of the S-matrix. The difficulty lies in that currently it is only known how to quantize the string theory in (generalized) light-cone gauge, where manifest Lorentz invariance is lost, and where it is unclear how to implement crossing symmetry. What Janik did was assume the existence of an Hopf algebra structure, underlying the long-range spin chain, which allowed him to implement the crossing symmetry algebraically. Kinematically, crossing is implemented through the simple transformation

$$x^{\pm} \to \frac{g^2}{x^{\pm}} \tag{2.5.43}$$

while the overall scalar factor $S^0_{12} = S^0_{12}(x^{\pm}_1, x^{\pm}_2)$ that was undetermined by the algebraic construction of the S-matrix must satisfy the constraint

$$1 = S^0_{12} S^0_{1\bar{2}} \frac{x_1^-}{x_1^+} \frac{x_2^+ - x_1^-}{x_2^- - x_1^-} \frac{1 - g^2/x_1^+ x_2^+}{1 - g^2/x_1^+ x_2^-} \ , \tag{2.5.44}$$

where

$$S^0_{1\bar{2}} \equiv S^0_{12}\left(x_1^{\pm}, \frac{g^2}{x_2^{\pm}}\right) \ . \tag{2.5.45}$$

These expressions contain hidden subtleties: If one performs a crossing transformation on the second particle in equation (2.5.44), the left hand side is obviously invariant, while the right hand side is not if one lets $S^0_{1\bar{2}}$ transform to S^0_{12} under such a transformation. The reason is that a double crossing is not an identity transform, but rather a change of sheet of the Riemann surface which S^0_{12} is defined on.

In [112] Beisert, Hernandez and Lopez then produced an all-loop proposal at strong coupling for the dressing factor, which satisfied crossing symmetry. Crossing was implemented separately for odd and even loop orders, and for odd orders only the one-loop HL correction to the phase entered, satisfying crossing by itself, as shown earlier in[111]. For even orders things are more complicated, and one receives contributions from all even orders. The expression for the dressing phase in [112] is perturbative, giving closed expressions for the phase at each order. In [113] a useful integral expression for the BHL dressing factor was then given.

A proposal for the weak coupling expansion of the dressing phase was given by Beisert, Eden and Staudacher in [87]. Based on properties of the digamma function, it was suggested that to obtain the weak coupling expansion from the strong coupling expansion, one should set

$$c_{r,\,s}(g) = -\sum_{n=1}^{\infty} c_{r,s}^{(-n)} g^{1+n} \ , \tag{2.5.46}$$

where $c_{r,\,s}(g) = g^{2-r-s}\beta_{r,\,s}(g)$, for the coefficients appearing in (2.5.41). This weak/strong continuation was then shown to be fully consistent in [114]. Having obtained a weak coupling expansion for the dressing, Beisert, Eden and Staudacher then studied how the ES equation, introduced in section 2.5, for the cusp anomalous dimension is modified by its introduction.

The solution of the corrected equation, known as the BES equation, provides the cusp anomalous dimension to all orders. Curiously, one can obtain it by performing the substitution

$$\zeta(2n+1) \to i\zeta(2n+1) \tag{2.5.47}$$

in the expansion of the cusp anomalous dimension obtained from the ES equation (odd arguments of the zeta function always appear in pairs so that the final result is real). And the BES equation itself can in fact be obtained from the ES equation by flipping a spin in the Kernel and iterating once! This simplicity would seem to imply that there exists some simple explanation for the dressing factor.

The BES equation implies $\beta = \zeta(3)$ in equation (2.5.16), in agreement with the four-loop amplitude calculation of the cusp anomalous dimension [8]. Curiously, and despite having been the basis of the BDS Ansatz, which in turn is a key component of the ABA, BMN scaling is therefore violated starting from four loops [8], since it is equivalent at four loops to a vanishing β. This means that previous all-loop expressions for BMN operator anomalous dimensions are incorrect starting from four loops.

Further justification for this all-loop BHS/BHL dressing factor was subsequently produced. The four-loop dressing coefficient β was obtained directly by a four-loop calculation in the $SU(2)$ sector in [115], and perhaps more importantly, after solving the difficult problem of expanding the BES equation at strong coupling [116], the equation was consistent with the string theoretical calculations displayed in (2.5.19), and also with direct evaluations of the cusp anomalous dimension performed by calculating Wilson lines with cusps at strong coupling [117].

With the all-loop dressing factor the final piece of the Asymptotic Bethe Ansatz had been put in place. In the asymptotic limit, where the length of the spin chain is sufficiently long, all evidence is in favor of the correctness of the ABA. Generalizing the cusp anomalous dimension, perhaps the most non-trivial check to date is the calculation of the "generalized scaling function" $f(g,\, j)$, giving the smallest anomalous dimension in the $SL(2)$ sector, in the limit

$$M \to \infty\,, \quad L \to \infty\,, \quad j \equiv \frac{L}{\log M} = \text{fixed}\ . \tag{2.5.48}$$

In [118], an integral equation, generalizing the BES equation, was derived from the ABA giving this quantity to all orders, in perfect agreement with subsequent strong coupling calculations [119].

[8]Even though it is valid in the string theory at strong coupling.

2.6 Wrapping and finite size

Despite the success of the Asymptotic Bethe Ansatz, as we have mentioned before, it is only valid for "asymptotically" long spin chains. For a general operator the action of the dilatation operator will include "wrapping contributions", neglected by the ABA. The name of these contributions stems from their origin as those Feynman diagrams that "wrap" around the entire spin chain, and were suggesting as early as [56] as a possible explanation for the three-loop discrepancy between gauge and string theory (even though it was later understood that the dressing factor cures this discrepancy and that wrapping is delayed until four loops). In the planar limit, the range of any given interaction can at most increase by one unit per order of perturbation theory, but wrapping effects will appear eventually. A naive estimate (which is exact for the $SU(2)$ sector) indicates that they start at order g^{2L}, where L is the length of the spin chain. At first, however, it was not even clear that the ABA failed to incorporate wrapping, since it had not been derived at higher loops directly from Feynman diagrams. But as mentioned in section 2.5, this was definitely shown to be the case in [12], by demonstrating that the ABA failed to satisfy the constraints imposed by BFKL, and was therefore missing a piece, starting from four loops.

On the string theory side it was mentioned in section 2.4.3 that the setup corresponding to the ABA consisted in having an infinite light-cone momentum, in light-cone gauge, the so-called de-compactifying limit, in which the world-sheet becomes a plane. Introducing wrapping effects on the gauge theory side therefore corresponds to having a finite light-cone momentum, giving a finite radius for the world-sheet cylinder. For this reason, the string-theoretic "dual" of wrapping effects are called "finite-size" effects. The failure of the string Bethe Ansatz to incorporate such effects was studied in [120], while they had also appeared in a number of exact solutions of the semi-classical string [121]. Characteristic of the finite-size effects are that they are exponentially suppressed in the radius of the world-sheet, or in other words in the length of the spin chain.

Incorporating wrapping/finite-size effects into the AdS/CFT integrable model turned out to be a formidable challenge. This work was initiated by Ambjorn, Janik and Kristjansen in [122], by viewing the light-cone string sigma model as an integrable quantum field theory on a plane, that is then compactified to a cylinder. The logic was that for relativistic quantum field theories there exists a well-defined procedure, developed by Lüscher [123], for including finite-size corrections, which could then possibly be extended to the case of the non-relativistic light-cone string sigma model.

Let us consider the calculation of the mass $m(L)$ of a state of the relativistic field theory living on a cylinder of circumference L. Basically, the Lüscher corrections take into consideration virtual particles propagating around the cylinder, and take the form

$$m(L) = m(L = \infty) + \Delta m_\mu(L) + \Delta m_F(L) \ , \tag{2.6.1}$$

where the first correction is called the μ-term, and corresponds to a particle splitting into two on-shell virtual particles that travel around the cylinder and then recombine, while the second term is the F-term, giving the contribution from a virtual particle loop wrapping around the cylinder. For a single mass scale m, in $1+1$ dimensions, these corrections take the form [124]

$$\frac{\Delta m_\mu(L)}{m(\infty)} = -\frac{\sqrt{3}}{2} \sum_{b,\,c} M_{abc} (-i) \mathrm{Res}_{\theta = 2\pi i/3} S_{ab}^{ab}(\theta) e^{-\frac{\sqrt{3}}{2} mL} \ , \tag{2.6.2}$$

$$\frac{\Delta m_F(L)}{m(\infty)} = -\int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-mL\cosh\theta} \cosh\theta \sum_b \left(S_{ab}^{ab}(\theta + i\pi/2) - 1 \right) \ , \tag{2.6.3}$$

where $S_{ab}^{ab}(\theta)$ is the infinite volume S-matrix and $M_{abc} = 1$ if c is a bound state of a and b, and vanishes otherwise, restricting the sum to precisely the on-shell virtual particles mentioned above. The integral in the F term takes the form

$$\int_{-\infty}^{\infty} dp\, e^{-EL} \sum (S - 1) \ , \tag{2.6.4}$$

since the energy and momentum are parameterized, in terms of rapidity as

$$E = m\cosh\theta \ , \quad p = m\sinh\theta \ . \tag{2.6.5}$$

As a guidline for extending the Lüscher approach to the non-relativistic case, Ambjorn, Janik and Kristjansen focus on the Thermodynamic Bethe Ansatz. For integrable relativistic field theories this method gives the exact finite-size correction, and therefore permits a derivation of the Lüscher corrections. The application of the TBA to finite-size effects was first performed by Zamolodchikov [125] and is based on the following argument: Consider the theory on a torus of circumferences L and R, where $R \to \infty$ is taken to be the time coordinate. The ground state energy is then given by

$$E_0(L) = -\frac{1}{R} \log Z \ , \tag{2.6.6}$$

where Z is the euclidean partition function. However, if we instead take R as the space coordinate and L as the time, this is simply the free energy $F = E - TS$ of the system with infinite length and temperature $1/L$. And since the length is infinite in this case, we can calculate possible configurations of excitations with the ordinary Bethe Ansatz. Minimizing the free energy F in the space of such configurations then gives the ground state energy of the original system. In principal one must thus solve a system of equations consisting, on the one hand, of the variational equation stemming from minimizing F, and on the other hand, of the Bethe Ansatz equations in the thermodynamic limit (where the number of excitations M are taken to infinity such that the mean density $\alpha = M/L$ remains finite). Such a system of equations had already appeared in earlier works, starting with [126], but then with the intent of actually finding the free energy for a system at finite temperature.

Fortunately, if one is only interested in calculating the free energy F, things simplify. For example, in the case of a theory with a single particle, in terms of an auxiliary function $\epsilon(\theta)$, the equation for the minimization of F takes the form

$$\epsilon(\theta) = LE_{TBA}(\theta) - (\phi * L)(\theta) \ , \tag{2.6.7}$$

where E_{TBA} is the energy of the excitations (simply $m \cosh\theta$ in the relativistic case), as seen in the infinite volume theory, $\phi(\theta)$ is given in terms of the S-matrix of the Bethe Ansatz as

$$\phi(\theta) = -i\frac{d}{d\theta}\log S(\theta) \ , \tag{2.6.8}$$

$$L(\theta) = \log(1 + e^{-\epsilon(\theta)}) \ , \tag{2.6.9}$$

and $\phi * L$ stands for the convolution

$$\int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\phi(\theta - \theta')L(\theta') \ . \tag{2.6.10}$$

The free energy, or equivalently the ground state energy, can then be obtained directly, without having to solve the Bethe Ansatz equations for the root density, as

$$E_0 = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} p'_{TBA}(\theta)L(\theta) \ . \tag{2.6.11}$$

Strictly speaking, the above analysis applies to the finite-size ground state $E_0(L)$. However, it was noted in [127] that the energies of the excited states can be obtained by analytically continuing the TBA equations. And the leading contributions to the energies obtained in this way coincide with Lüscher's formulae.

Ambjorn et al then take inspiration in the space-time exchange picture implied by the TBA when trying to generalize the Lüscher correction to the non-relativistic case. The idea is to take as the momentum and energy in the F-term formula (2.6.4) the momentum p_{TBA} and energy E_{TBA} obtained by Wick rotating the original momentum and energy. If q and $E(q)$ denote the original quantities, one sets

$$p_{TBA} \equiv Q = -iE(q) \ , \tag{2.6.12}$$

and

$$E_{TBA} = -iq(Q) \ . \tag{2.6.13}$$

Applying this to the dispersion relation of $\mathcal{N}=4$, given, as before, by

$$E(q)=\sqrt{1+16g^2\sin^2\frac{q}{2}}\ , \tag{2.6.14}$$

inversion gives the exponential factor in the F-term as

$$e^{-2L\text{arcsinh}\left(\frac{\sqrt{1+p_{TBA}^2}}{4g}\right)}\ , \tag{2.6.15}$$

which is promising since at weak coupling the arcsinh has a large argument and behaves like a logarithm, giving a leading behavior of the order g^{2L}, as expected from wrapping corrections. The strong coupling behavior could also reproduce expectations from finite-size corrections. One should keep in mind, however, that in the non-relativistic case, the theory obtained by performing this double Wick rotation, dubbed the mirror theory, is in general not equivalent to the original theory and has in particular a different set of Bethe equations, complicating the application of the TBA. The precise study of this mirror theory was initiated in [129].

The results of [122] were rather qualitative, so more direct evidence in favor of the modified Lüscher approach was needed. The first success was that of calculating the leading finite-size correction to the dispersion relation of the giant magnon, which is the string state, constructed in [128], corresponding to a spin chain magnon of finite momentum p. In [130], Janik and Lukowski where able to identify the μ-term of the Lüscher correction at strong coupling, with the finite-size correction

$$E-J=\frac{\sqrt{\lambda}}{\pi}\sin\frac{p_{ws}}{2}\left(1-\frac{4}{e^2}\sin^2\frac{p_{ws}}{2}e^{-\frac{2\pi J}{\sqrt{\lambda}\sin\frac{p_{ws}}{2}}}\right)\ , \tag{2.6.16}$$

where p_{ws} is the world-sheet momentum, obtained earlier in [131] through an analysis of the semi-classical string [9], and J plays the role of the circumference of the world-sheet. The evaluation of the μ-term involved a Borel summation over an infinite number of coefficients of the dressing phase, necessary in order to produce the non-trivial $\frac{1}{e^2}$ coefficient of (2.6.16). As such the calcuation of Janik and Lukowski also provided an all-order check for the BHL/BES dressing phase.

Instead of reproducing the full calculation of [130], let us simply give a shorter motivation (also given in [130]) for how the μ-term can give the correct exponential behavior.

[9] Actually, the generalized light-cone gauge has a gauge parameter a which enters into the finite-size correction to the dispersion relation of [131]. However, in order to have periodicity in the momenta, and a behavior compatible with the gauge theory, one is forced to choose $a=0$, which corresponds to the formula displayed here. The issue is not completely clear, however. Although $a=0$ seems to be the most natural choice from the viewpoint of AdS/CFT, Janik and Lukowski were able to reproduce the a-dependence of [131] from the Lüscher formula. On the other hand, the authors of [132] claim that the dispersion relation becomes independent of a if one treats the gauge fixing differently.

From a generalization of (2.6.2) for non-zero p, one finds that the exponential in the μ-term can be rewritten

$$e^{-J\cdot\operatorname{Im} p_c}\ , \tag{2.6.17}$$

when the particle of momentum p splits into on-shell constituents with momenta p_c and $p-p_c$. Since the virtual particles are on-shell, one has

$$\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}=\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p_c}{2}}+\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p-p_c}{2}}\ . \tag{2.6.18}$$

At strong coupling this gives, perturbatively

$$p_c=p+\frac{2\pi i}{\sqrt{\lambda}\sin\frac{p}{2}}\ , \tag{2.6.19}$$

which, when introduced in (2.6.17) gives

$$e^{-J\cdot\operatorname{Im} p_c}=e^{-\frac{2\pi J}{\sqrt{\lambda}\sin\frac{p}{2}}}\ , \tag{2.6.20}$$

which is precisely the exponential terms appearing in the finite-size correction (2.6.16) to the magnon dispersion relation. Further articles calculating the μ and F-terms at strong coupling are [133], in which it is proven that the F-term is subleading compared to the μ-term at strong coupling, explaining why the latter gives the correct correction to the dispersion relation.

Despite the conceptual importance of the reproduction of the strong coupling finite-size correction to the dispersion relation, the truly interesting development came in [134] , where Bajnok and Janik applied the Lüscher approach to calculate the first wrapping correction to the Konishi operator (or more precisely its $SL(2)$ sector descendent) anomalous dimension at weak coupling. Amazingly, the result (including the asymptotic part),

$$\gamma_{\text{Konishi}}=4+12g^2-48g^4+336g^6+g^8(-2496+576\zeta(3)-1440\zeta(5)) \tag{2.6.21}$$

coincided with an impressive four-loop super-Feynman diagram calculation by Fiamberti et. al. [135], and the later direct component evaluation [136] by Velizhanin. In order to arrive at (2.6.21), Bajnok and Janik had to generalize the one-particle Lüscher correction to multi-particle states. In contrast to the strong coupling calculation of the finite-size dispersion relation, a complication which enters at weak coupling is the need to include an infinite number of bound states when summing over the virtual states propagating around the cylinder.

At this point it was not clear whether one should sum over the bound states of the original theory, studied in [137], or the bound states of the mirror theory. In the original theory, physical boundstates with a finite number of magnon constituents only exist for the $SU(2)$ sector, while in the mirror theory the boundstates live in the $SL(2)$ sector. It turned out that the latter gave sensible results, but in order to obtain them the S-matrix for scattering of such bound states with fundamental magnons had to be calculated (this calculation had been initiated in [129]). The resulting formula for the M-particle energy, including the Lüscher correction is

$$E(L) = \sum_{k} \epsilon(p_k) - \sum_{j,\,k} \frac{d\epsilon(p_k)}{dp_k} \left(\frac{\delta BY_k}{\delta p_j}\right)^{-1} \delta\Phi_j - \\ - \int_{-\infty}^{\infty} \frac{d\tilde{p}}{2\pi} \sum_{a_1,\,\ldots,\,a_M} (-1)^F \left[S^{a_2 a}_{a_1 a}(\tilde{p},\, p_1) S^{a_3 a}_{a_2 a}(\tilde{p},\, p_2) \cdots S^{a_1 a}_{a_M a}(\tilde{p},\, p_M)\right] e^{-\tilde{\epsilon}_{a_1}(\tilde{p})L} \,. \quad (2.6.22)$$

Here, $\epsilon(p_k)$ is the ordinary magnon dispersion relation, while $\tilde{\epsilon}_{a_1}$ is the mirror dispersion relation for particle type a_1. The index a stands for an ordinary magnon in the $SL(2)$ sector. The momenta p_k are given as the solution of the infinite volume Bethe Ansatz equations, or equivalently, the roots of the Baxter Q-function. The first term of (2.6.22) is the infinite-volume energy, the second term incorporates the finite-volume corrections to the momenta of the particles, while the last term is the analogue of the F-term, having an interpretation of summing contributions from virtual particles which propagate around the cylinder, with the exponential term playing the role of the propagator, and scattering with all M particles of the multi-particle state along the way. At lowest order the last term will dominate over the second term and the wrapping correction will be given by the F-term. This should be compared to the case of strong coupling, where it was the μ-term which was responsible for the finite-size corrections. By contrast, the μ-term is not present at weak coupling.

Evaluating (2.6.22) for the special case of $M = 2$ gives the Konishi operator dimension (2.6.21). The only subtlety is that it is not obvious which value to choose for L. Applying the magnon bound state dispersion relation one finds that the exponential term for a Q-magnon bound state is, to lowest order,

$$\frac{4^L g^{2L}}{(Q^2 + q^2)^L} \,. \quad (2.6.23)$$

For $L = 4$ the exponential would seem to give a wrapping correction appearing at the correct perturbative order. However, the S-matrix factors may also contribute factors of the coupling, and it turns out that the correct prescription is to choose $L = 2$. This coincides with the length of the Konishi descendent in the $SL(2)$ sector, and is also natural from the point of view of the string worldsheet, since in the light cone gauge the simplest choice is to identify the length of the string with the R-charge J. In [94], Bajnok, Janik and Lukowski extended the calculation of the Konishi dimension to all $SL(2)$ twist-two operators, basically by evaluating (2.6.22), using (2.6.23) for the exponential term, for a number of different values of M, and thereby determining the coefficients for a basis of harmonic sums. In the process, a convenient explicit form for the wrapping part of (2.6.22) was given. It was presented for $L = 2$, but we show it here with the L-dependence re-introduced since we will make use of it later:

$$\gamma^{(L)}_{4,\text{wrap}}(M) = -4^L g^{2(L+2)} (\gamma_1^{(L)}(M))^2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{T_M(q,\,Q)^2}{R_M(q,\,Q)} \frac{1}{(q^2+Q^2)^L} \ , \tag{2.6.24}$$

where

$$\begin{aligned} R_M(q,\,Q) =& P_M\left(\frac{1}{2}(q - i(Q-1))\right) P_M\left(\frac{1}{2}(q + i(Q-1))\right) \cdot \\ & P_M\left(\frac{1}{2}(q + i(Q+1))\right) P_M\left(\frac{1}{2}(q - i(Q+1))\right) \ , \end{aligned} \tag{2.6.25}$$

$$T_M(q,\,Q) = \sum_{j=0}^{Q-1} \left[\frac{1}{2j - iq - Q} - (-1)^M \frac{1}{2(j+1) - iq - Q}\right] P_M\left(\frac{1}{2}(q - i(Q-1)) + ij\right) \ , \tag{2.6.26}$$

$\gamma_1^{(L)}(M)$ is, as before, the coefficient of the one-loop energy, and where P_M is the Baxter Q-function, entering the Baxter equation, and given by

$$2i \prod_{i=1}^{M} (u - u_i) \ . \tag{2.6.27}$$

The constant pre-factor does not matter in the case of the Baxter equation, but is relevant in this case.

Evaluating (2.6.24) for $L = 2$ and several M, Bajnok, Janik and Lukowski were able to obtain the wrapping correction for twist-two operators as

$$\begin{aligned} \gamma^{(2)}_{4,\,\text{wrap}}(M) =& -640 S_1^2 \zeta(5) - 512 S_1^2 S_{-2} \zeta(3) + \\ & + 256 S_1^2 \left(-S_5 + S_{-5} + 2S_{4,1} - 2S_{3,-2} + 2S_{-2,-3} - 4S_{-2,-2,1}\right) \ . \end{aligned} \tag{2.6.28}$$

Together with the ABA part of the twist-two four-loop spectrum, which is rather lengthy and can be found in [12], this gives the complete expression for the four-loop spectrum. Taking into account the amount of work that goes into the direct evaluation of a single one of these operator dimensions, such as the Konishi dimension, we can clearly see the strength of the integrability based approach. Of course, this strength relies on the correctness of (2.6.28), but all the available information on the four-loop twist 2 anomalous dimensions suggest that this formula is indeed correct. Apart from the Konishi dimension, Velizhanin had already calculated the coefficient of the $\zeta(5)$ corresponding to $M = 4$ [138]. Furthermore, the wrapping correction (2.6.28) vanishes in the limit $M \to \infty$, in accordance with the ABA correctly producing the cusp anomalous dimension in this limit, and more importantly, it was also shown in [94] that analytically continuing to $M = -1$ gives a behavior fully consistent with the NLLA BFKL constraint (2.5.22). The full four-loop twist-two spectrum including the wrapping correction thus passes the test that the Asymptotic Bethe Ansatz failed in [12].

The Lüscher approach has since had continued success. It was applied in [139] to obtain the five-loop twist-three spectrum (consistent with a direct calculation in [140]), in [141] to calculating the Konishi dimension at five loops, a calculation which was extended to all twist-two operators at five loops in [13] (consistent with constraints from BFKL), and was finally used to produce the six-loop twist-three spectrum in [142]. In the next chapter, we will also describe some applications of this method to the β-deformed theory.

Lüscher's approach is not a complete solution, though, since it provides corrections, but no exact expression for the spectrum of anomalous dimensions. For this reason, it has been a major goal to find a model providing such an exact expression. Excitingly, this problem seems close to be solved. In [143], taking inspiration in [144], a system of functional equations known as a Y-system, was conjectured to produce the exact spectrum of planar AdS/CFT, and seems to pass all available checks (see for example [140, 145, 146, 147]), although some doubts have been put forward [148]. The full Thermodynamic Bethe Ansatz has also been developed in [149], providing an alternative (and in fact it seems that the Y-system can be derived from the TBA). Furthermore, these models place some of the Lüscher calculations on a firmer footing, providing alternate derivations for the four-loop twist-two spectrum and the five-loop Konishi anomalous dimension [150]. There are, however, some subtleties that still need to worked out, and it is still not entirely clear if the slightly different versions of the models appearing in the literature are equivalent.

Whatever form the final integrable system takes, what remains is of course to derive it. This seems to be more tractable from the viewpoint of the string theory, since the TBA is naturally formulated in terms of the string sigma-model. In order to provide a proof of the equality of the planar spectra of the gauge and string theory, however, one must also derive the same system from the gauge theory, a problem which seems much more difficult.

2.7 On the interplay between string and gauge theories.

The interplay between string and gauge theories have proved to be very important in the construction of the models now in place for calculating the spectra of the theories. To begin with, it would have been very difficult, if not impossible, to construct the ABA together with the TBA and the Lüscher corrections from the point of view of the gauge theory without string theory input, for the following reasons

1. The BDS ansatz would have been difficult to make without the notion of BMN scaling. The algebraic construction of Beisert could still have been done, but without the string formulae, it would probably not have been written in terms of the $x^{\pm}$ variables, making it difficult to obtain the "trivial" scalar factor S_0. Experimentally, BMN scaling could of course have been discovered, but it would have to have been done before the completion of the calculation of the four-loop cusp anomalous dimension, which violates BMN scaling.

2. Without the notion of crossing symmetry, coming from the relativistic string worldsheet, and the calculation of the HL phase, it would seem difficult to find the current all-loop dressing phase.

3. The Lüscher corrections would probably not have been proposed without the notion of an underlying two-dimensional field theory.

But it is not certain that sufficient progress would be made if we only had access to the string theory, either. Without the Heisenberg model of the one-loop gauge theory, it is not as likely that a discretized version of the continuum string Bethe equations, such as AFS, would have been put forward (or received as much attention). And it would probably not have been written in terms of an S-matrix proportional to the Heisenberg model S-matrix. And from a perturbative string theory point of view, in the current framework, the one-loop correction to the dressing phase is all that is needed to satisfy crossing to all odd perturbative orders, but if a different all-loop ansatz would have been chosen, all-loop dressing corrections would have been needed also for the odd perturbative orders. It could also happen that a different solution of the crossing equations be chosen. Still, it would seem that the string theory is the most likely candidate for a derivation of the complete spectrum of energies.

Chapter 3

A symmetry of the marginally deformed spin chain and the four-loop dilatation operator

In this chapter, our focus will be that of the marginal β-deformation of $\mathcal{N} = 4$ Super Yang Mills. Our main goal, however, will be to use this theory in order to better understand wrapping corrections in the original $\mathcal{N} = 4$ theory, and we will therefore not give it as complete a treatment as we did for the undeformed theory in the previous chapter. In the next section we will define the β-deformed SYM theory, and discuss how integrability carries over from the undeformed theory. We will then present a conjecture relating the integrable models for different values of the deformation in section 3.2, together with an explicit calculation of the single magnon energy in favor of this conjecture. We then end the chapter with section 3.3, discussing some developments that have recently appeared in the literature together with an outlook on possible directions for future work.

3.1 Marginally deformed $\mathcal{N} = 4$ SYM and integrability

With its (probable) planar integrability, the $\mathcal{N} = 4$ theory is indeed interesting. However, with so much symmetry present, it is not a particularly realistic theory, and one would like to find ways to reduce the amount of symmetry present, while keeping some of its more desirable properties. In [14], Leigh and Strassler initiated the study of the so-called marginal deformations of the $\mathcal{N} = 4$ theory, which was then further developed in [151]. The $\mathcal{N} = 4$ Lagrangian can be written in terms of $\mathcal{N} = 1$ superfields, in which case the superpotential takes the form

$$W = g\mathrm{Tr}\left([X, Y]Z\right) \ , \tag{3.1.1}$$

where X, Y and Z are chiral superfields. There is then a two-parameter family of marginal deformations of this superpotential, given as

$$W_{marg} = a\mathrm{Tr}\left(XYZ - qYXZ + \frac{\lambda}{3}\left(X^3 + Y^3 + Z^3\right)\right) \ , \tag{3.1.2}$$

having the property that they preserve $\mathcal{N} = 1$ supersymmetry and conformal symmetry.

But the thing we really like about $\mathcal{N} = 4$ is its integrability, so it seems logical to limit ourselves to those deformations that preserve some integrability in the planar limit. This problem was studied for the first time by Roiban in [152], where it was found that the q-deformation of $\mathcal{N} = 4$, which is obtained by setting $\lambda = 0$, had, at one-loop, an integrable subsector, corresponding to the $SU(2)$ sector of the undeformed theory (recovered by setting $q = 1$). This sector still consists of operators of the form

$$\mathrm{Tr}ZZWZ \cdots ZWZ \ , \tag{3.1.3}$$

and since our main results will be related to it we will continue calling it the $SU(2)$ sector for brevity, despite it only having this symmetry in the undeformed case (In the q-deformed theory the R-symmetry is broken to $U(1)^3$).

In general, q is an arbitrary complex number, which we can write as

$$q = \frac{e^{2\pi i\beta}}{\alpha} \ , \tag{3.1.4}$$

for real α and β. In [152] it was also seen that if one tried to extend the integrable model to the sector of three scalars, it was necessary to choose $\alpha = 1$, or, in other words, q had to be a phase. This case, which will be our main focus, is called the β-deformed theory, with the real number β identifying the deformation. Apart from the obvious symmetry $\beta \to \beta + 1$, the spectrum of the theory also has the symmetry $\beta \to -\beta$. This later symmetry is related to parity transformations (interchanging the order of the sites of the spin chain), since a parity transformation acting on an eigenstate of the dilatation operator at deformation β produces an eigenstate at $-\beta$ (and parity is thus broken for general β).

This issue was studied in further detail by Berenstein and Cherkis in [153], were it was confirmed that only for the β-deformed theory was it possible to have integrability for the full theory at one loop. Of course, if we are searching for complete integrability of the theory we must have integrability for the entire algebra at one loop as a prerequisite.

In the article by Berenstein and Cherkis, it was also explained, that in the $SU(2)$ sector, the β-dependence on the Hamiltonian could be removed [1], recovering the XXX

[1]This statement is actually valid for an arbitrary $q - deformation$, with the result that the XXZ Hamiltonian is obtained, having a quantum symmetry of $SU(2)_\alpha$.

Hamiltonian of the undeformed theory, by performing a change of basis. If we introduce the standard spin chain basis, as we did in the case of the $SU(2)$ sector in the undeformed theory, by forming tensor products of states such as $|0\rangle_k$, representing a Z field at site k, and $|1\rangle_k$ for a W field at the same site, the Hamiltonian, as obtained by calculating the action of the dilation operator or the mixing matrix, is given (up to a constant pre-factor) by

$$H = -\frac{1}{2}\sum_{i=1}^{L}\left[\left(\sigma_l^z\otimes\sigma_{l+1}^z - 1_l\otimes 1_{l+1}\right) + \cos 2\pi\beta\left(\sigma_l^x\otimes\sigma_{l+1}^x + \sigma_l^y\otimes\sigma_{l+1}^y\right) + \right.$$

$$\left. + \sin 2\pi\beta\left(\sigma_l^x\otimes\sigma_{l+1}^y - \sigma_l^y\otimes\sigma_{l+1}^x\right)\right] \ . \tag{3.1.5}$$

We can here clearly see how the last term breaks the parity invariance of the Heisenberg XXX case, given by equation (2.3.10), an expression that is recovered at $\beta = 0$ by using that the permutation operator can be rewritten in terms if the Pauli matrices through

$$P_{i,\,i+1} = \frac{1}{2}\left(\mathbb{1}_{i,\,i+1} + \sigma_i^z\otimes\sigma_{i+1}^z + \sigma_i^+\otimes\sigma_{i+1}^- + \sigma_i^-\otimes\sigma_{i+1}^+\right) \ . \tag{3.1.6}$$

Acting on two adjacent sites, it can easily be checked that the only β-dependent matrix elements of (3.1.5) are

$$\langle 1\,0|H|0\,1\rangle = -e^{-2\pi i\beta} \quad \text{and} \quad \langle 0\,1|H|1\,0\rangle = -e^{2\pi i\beta} \ . \tag{3.1.7}$$

This dependence can be removed by the position-dependent change of basis

$$|0\rangle_k = |\tilde{0}\rangle_k \quad \text{and} \quad |1\rangle_k = e^{-2\pi i k\beta}|\tilde{1}\rangle_k \ , \tag{3.1.8}$$

and written in terms of the basis $\left\{|\tilde{0}\rangle_k\,,\,|\tilde{1}\rangle_k\right\}$, the Hamiltonian will be β-independent. This does not mean that the spectrum of the theory is β-dependent, however, since the periodic boundary conditions of the original basis will turn into twisted boundary conditions after having performed (3.1.8). More precisely,

$$|\tilde{1}\rangle_L = e^{2\pi i\beta L}|\tilde{1}\rangle_0 \ . \tag{3.1.9}$$

For practical applications, the picture of the undeformed Hamiltonian acting on a spin chain with twisted boundary conditions is often preferable. For example, the Bethe Ansatz, can be obtained rather directly from the Ansatz of the original theory. The vacuum state can still be taken as $\mathrm{Tr}Z^L$, as this state was shown in [154] to be protected by supersymmetry, and therefore has 0 anomalous dimension, and since it is unaffected by the change of basis (3.1.8). With the intuitive idea in mind that the Bethe Ansatz represents a periodicity condition for the magnon wavefunctions, we obtain from (3.1.9) and (2.3.22), the one-loop twisted Bethe Ansatz

$$e^{-2\pi i\beta L}\left(\frac{u_k + i/2}{u_k - i/2}\right)^L = \prod_{j\neq k=1}^{M}\frac{u_k - u_j + i}{u_k - u_j - i} \ , \tag{3.1.10}$$

while the formula for the magnon energies in terms of the rapidities u_k,

$$E_k = \frac{1}{u_k^2 + 1/4} \tag{3.1.11}$$

is unaltered.

The cyclicity constraint must also be modified when twisted boundary conditions are used. In the original basis, the trace implies an invariance under shifts by one site along the spin chain. In the new basis, such a shift produces an additional $e^{2\pi i\beta}$ factor for each $|\tilde{1}\rangle$ in the chain. Since an M magnon state consists of a sum over states with M insertions of such factors, the cyclicity constraint becomes

$$\prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2} = e^{2\pi i\beta M} \ . \tag{3.1.12}$$

3.1.1 The string dual and higher loops

In $\mathcal{N} = 4$, an important guide in constructing the all-loop integrable model was the $AdS_5 \times S^5$ string theory. In order to try to construct a similar model in the case of the β-deformed theory, one would thus like to have a string dual to fall back on. At first, its construction proved to be quite challenging, and was only known for some special cases, such as when q is an n-th root of unity, in which case the string background becomes

$$AdS_5 \times S^5/\mathbb{Z}^n \times \mathbb{Z}^n \ , \tag{3.1.13}$$

or for perturbations around the undeformed case [155]. The problem was finally solved in [156] by Lunin and Maldacena, where the deformed background was constructed by a so-called TsT transformation. This transformation consists in performing a T-duality, followed by a shift, proportional to the deformation β, in an angular coordinate on the S^5, after which one performs another T-duality.

Making use of the construction of the deformed background in terms of this TsT transformation, Frolov showed [157], in a similar way that as been done for the undeformed theory [158], that the corresponding string sigma model was classically integrable by deriving a Lax connection from the classical equations of motion. At the same time, in [159], Frolov, Roiban and Tseytlin explained that the sigma model of the deformed theory could be obtained from the original sigma model by twisting its boundary conditions, similarly to how the deformed one-loop spin chain can be described in terms of the undeformed Hamiltonian by simply changing the boundary conditions. Furthermore, by studying the structure of the gauge theory Feynman diagrams they also drew the conclusion that the two-loop extension of the $SU(2)$ spin chain had the same property. In fact, the thermodynamic limit of the Bethe equations for this two-loop spin chain coincided to this order with the integral Bethe equations for the $SU(2)$ sector of the twisted sigma model.

So it would seem that the planar β-deformed theory could be completely integrable, with its integrability properties derived from twisting the original theory. This analysis was performed for the all-loop Asymptotic Bethe Ansatz by Beisert and Roiban in [160]. In this article it was also mentioned that the twisted theory should be integrable at the quantum level, since the calculation performed by Berkovits [50] should carry over to the twisted theory with only minor modifications.

As in the undeformed case, having the spectrum of asymptotically long operators under control, the next step is to introduce wrapping/finite-size corrections. On the gauge theory side, wrapping calculations were performed in [161, 162] (reviewed in [163]) for operators corresponding to $M = 2$ and $M = 1$ in the $SU(2)$ sector, were the latter case is non-trivial since in the β-deformed theory, the single magnon operator is no longer protected by supersymmetry and can thus acquire a non-vanishing anomalous dimension. This operator will be discussed further in section 3.2.1. Another calculation which we will also return to, on the string side, is that of the finite-size corrections of the magnon dispersion relation [164] by Bykov and Frolov. Apart from these works it was only very recently that progress has been made on wrapping corrections. We will mention some of these newer developments in section 3.3.

3.2 A symmetry of the twisted spin chain, and the $\beta = 1/2$ magnon

In this section we will present a conjecture for a symmetry, presented by the author in [165] , relating the $SU(2)$ spin chain spectrum, including wrapping, for different values of β. More precisely, we conjecture that for a spin chain of length L, the spectrum is invariant under the change

$$\beta \to \beta + \frac{n}{L} \ , \tag{3.2.1}$$

for abitrary integer n, if the cyclicity constraint is relaxed. Also, for some states, the symmetry will be respected even if we do not relax the cyclicity constraint. In general, however, it will be not be directly visible in the physical spectrum of the gauge theory. But it will still have important consequences for the physical spectrum, as we will see.

Here, we will discuss how the symmetry enters into the twisted Asymptotic Bethe Ansatz, and show that some previous results in the literature on wrapping/finite-size corrections are consistent with it. In the next section we will give some additional proof by evaluating the Lüscher correction for the single magnon anomalous energy at deformation $\beta = 0$, $p = \pi$, and comparing it to the physical magnon anomalous dimension at $\beta = 1/2$, for spin chain lengths $L = 4$, 6, and 8, as well as matching the coefficient of maximal trascendentality for all L. As an application, we note that the equality of these energies provides an efficient way of calculating the first wrapping correction to the physical magnon operator anomalous dimension at $\beta = 1/2$. One of our main interests in this symmetry, its consequences for the original $\mathcal{N} = 4$ theory, will then be discussed in section 3.2.2.

To begin with, we note that the one-loop Ansatz (3.1.10), and the magnon energy (3.1.11), are invariant under (3.2.1), since the deformation only enters through the factor $\exp(-2\pi i\beta L)$. The same will be true for the all-loop Ansatz, since the deformation is conjectured to enter in the same way at all orders [159, 160], as a consequence of the twisted boundary conditions. Thus, any set of rapidities solving the Bethe equations for deformation β will also do so for $\beta + n/L$, and vice versa.

This invariance can in fact be seen directly at the level of the asymptotic Hamiltonian. In section 2.3.2 we mentioned that in the $SU(2)$ sector, the undeformed Hamiltonian could be written as a linear combination of generalized permutations,

$$\{n_1, \ldots, n_k\} \equiv \sum_{l=0}^{L-1} P_{n_1+l,\, n_1+l+1} \cdots P_{n_k+l,\, n_k+l+1} \ , \tag{3.2.2}$$

where the $P_{i,\, i+1}$ permutes the spins at sites i and $i+1$. In the previous section, we displayed in equation (3.1.6) a standard representation for such permutations in terms of the Pauli matrices. The asymptotic deformed Hamiltonian can be obtained from the undeformed one by simply replacing all permutations (3.1.6) with deformed operators [153, 159, 161],

$$\boldsymbol{P}_{i,\, i+1} = \frac{1}{2}\left(\mathbb{1}_{i,\, i+1} + \sigma_i^z \otimes \sigma_{i+1}^z + e^{2\pi i\beta}\sigma_i^+ \otimes \sigma_{i+1}^- + e^{-2\pi i\beta}\sigma_i^- \otimes \sigma_{i+1}^+\right) \ . \tag{3.2.3}$$

The coefficients that multiply the generalized permutations do not change when going to the deformed theory. Now, making the change $\beta \to \beta + n/L$ produces additional phases multiplying the last two terms in (3.2.3), but which can be removed by performing a non-local change of basis, of the same form as (3.1.8). Normally, such a change of basis introduces an additional twist in the boundary conditions, but when the change in deformation is n/L, the twist becomes unity.

The only thing that can fail is the cyclicity constraint (3.1.12), restricting the spectrum to those state corresponding to gauge-invariant operators. We note that applying (3.2.1) transforms the RHS of (3.1.12) as

$$e^{2\pi i\beta M} \to e^{2\pi i\beta M} e^{2\pi i \frac{nM}{L}} \ . \tag{3.2.4}$$

One way to interpret this is to say that (3.2.1) shifts the momentum of the state[2]. If this

[2]This assumes that we are using the picture of a periodic spin chain, so that physical operators correspond to zero total momentum. Alternatively, if we eliminate the deformation dependence of the dispersion relation and the S-matrix by introducing twisted boundary conditions the momentum is not altered upon changing β, and the cyclicity constraint instead gives a β-dependent momentum condition for physical operators.

shift is not a multiple of 2π, the symmetry will relate a physical state at deformation β with an unphysical state at deformation $\beta + \frac{n}{L}$. For example, in the next section we will be interested in the special case $n = L/2$, giving the symmetry $\beta \to \beta + 1/2$, present for spin chains of even length. The shift (3.2.4) then implies that physical states with odd magnon number will be mapped to unphysical states of total momentum $p = \pi$.

In some cases the cyclicity constraint is not affected. For example, if $n = L/2$, the cyclicity constraint is invariant for an even magnon number. In particular, this means that the physical operator dimension, corresponding $L = 4$, $M = 2$, and giving the Konishi dimension at $\beta = 0$, should be invariant under the shift $\beta \to \beta + 1/2$. That this is the case for the asymptotic part of the anomalous dimension follows from our previous discussion, but interestingly, the first wrapping correction to this operator also satisfies the symmetry. In [161] the anomalous dimension of the two-impurity operators (For general β the Konishi dimension is split into two operator dimensions) were calculated for $L = 4$, with the result that the deformation enters through the function $\Delta = \cos(4\pi\beta)$. This is of course invariant under $\beta \to \beta + 1/2$.

Further evidence in favor of the symmetry arises from string theory. The first finite size correction to the magnon dispersion relation in the β-deformed theory was calculated in [164] at strong coupling. The correction is

$$\delta E = \frac{4\sqrt{\lambda}}{\pi e^2} \sin^3 \frac{\tilde{p}}{2} \cos \Phi \, e^{-\frac{2\pi L}{\sqrt{\lambda} \sin \tilde{p}/2}} \ , \tag{3.2.5}$$

with the deformation entering through a quantity $\cos \Phi$, where

$$\Phi = \frac{2\pi(n_2 - \beta L)}{2^{3/2} \cos^3 \frac{\tilde{p}}{4}} \ , \tag{3.2.6}$$

with n_2 being a winding number, which in order for the existence of a solution to the equations of motion, has to be chosen as $[\beta L]$, the integer nearest to βL. The quantity $\tilde{p}$ is the momentum when twisted boundary conditions are used, or if one prefers periodic boundary conditions is related to the periodic momentum p via

$$\tilde{p} = p + 2\pi\beta \ . \tag{3.2.7}$$

The momentum $\tilde{p}$ is invariant under the symmetry, and the phase (3.2.6) is so as well, but only because the winding has been set to $[\beta L]$. Different choices of the winding, such as a fixed number, would violate the symmetry for general $\tilde{p}$.

3.2.1 Equality of magnon energies

We will now give evidence, including the first wrapping correction, for the symmetry $\beta \to \beta + 1/2$ for $M = 1$ and even L, which according to (3.2.4) should relate a physical magnon at deformation β with a non-physical magnon of momentum $p = \pi$ at $\beta + 1/2$. More precisely, we will give evidence that the energy of the magnon of momentum $p = \pi$ in the undeformed theory is the same as the anomalous dimension of the physical magnon operator at $\beta = 1/2$.

To begin with, the asymptotic dispersion relation in the β-deformed theory is [159]

$$E = -1 + \sqrt{1 + 16g^2 \sin^2\left(\frac{\tilde{p}}{2}\right)}\ , \tag{3.2.8}$$

with $\tilde{p}$ given in (3.2.7). For a non-zero β-deformation, single-magnon operators

$$\mathrm{Tr}\left(WZ^{L-1}\right) \tag{3.2.9}$$

are not protected by supersymmetry and thus acquire non-zero anomalous dimension given by (3.2.8) with $\tilde{p} = 2\pi\beta$, as was confirmed by explicit calculation in [166]. Using the definition (3.2.7), physical operators thus correspond to $p = 0$ also in the β-deformed theory. And since $\tilde{p}$ is invariant under (3.2.1), the symmetry maps magnons to magnons of the same energy. In particular $\tilde{p} = \pi$ both corresponds to the $p = \pi$, $\beta = 0$ and the $p = 0$, $\beta = 1/2$ cases.

Let us now move on to the wrapping corrections, first studying the case $L = 4$ after which we move on to larger L. In [162], the calculation of the first wrapping contribution to the anomalous dimensions of the single magnon operators (3.2.9) in the β-deformed theory was presented. It was shown that the correction, appearing at order g^{2L}, takes the general form

$$\delta\gamma_L^\beta = -2Lg^{2L}\Big[\big(C_{L,0}(\beta) - C_{L,L-1}(\beta)\big)P_L - 2\sum_{j=0}^{\lfloor \frac{L}{2}\rfloor - 1}\big(C_{L,j}(\beta) - C_{L,L-j-1}(\beta)\big)I_L^{(j+1)}\Big]\ , \tag{3.2.10}$$

where

$$C_{L,j}(\beta) = -8\sin^2(\pi\beta)\cos\left[2\pi\beta(L-j-1)\right]\ , \quad P_L = \frac{2}{L}\binom{2L-3}{L-1}\zeta(2L-3)\ . \tag{3.2.11}$$

The integrals $I_L^{(j+1)}$ are presented explicitly up to nine-loops in [162]. When $\beta = 1/2$ one has

$$C_{r,j} = 8(-1)^{r-j} \tag{3.2.12}$$

and so

$$\delta\gamma_L^{1/2} = -16Lg^{2L}\left(1 - (-1)^{(L-1)}\right)\Big[P_L - 2\sum_{j=0}^{\lfloor \frac{L}{2}\rfloor - 1}(-1)^j I_L^{(j+1)}\Big]\ . \tag{3.2.13}$$

Interestingly, the wrapping contribution vanishes for all odd operator lengths when $\beta = 1/2$. Furthermore, using (see the appendix in [162])

$$I_4^{(1)} = \frac{1}{2}\zeta(3) + \frac{5}{2}\zeta(5)\ , \quad I_4^{(2)} = -\frac{3}{2}\zeta(3) + \frac{5}{2}\zeta(5)\ , \tag{3.2.14}$$

we get the first non-trivial wrapping correction to the operator of length four as

$$\delta\gamma_4^{1/2} = g^8\left(512\zeta(3) - 640\zeta(5)\right)\ . \tag{3.2.15}$$

We would like to compare this wrapping correction to the energy of the magnon of momentum π in the original $\mathcal{N} = 4$ theory. Fortunately, the formula (2.5.25) allows us to obtain precisely this from the physical spectrum. Strictly speaking, this equation is valid for the $SL(2)$ sector, but since the magnon dispersion relation should be universal, the first wrapping correction for this sector should coincide with that of the $SU(2)$ sector. The lengths of the spin chain in the different sectors will also be different, but at four loops the first wrapping correction appears for $L = 4$ in the $SU(2)$ sector, and for $L = 2$ in the $SL(2)$ sector, so it seems reasonable to expect that (2.5.25) indeed allows us to extract the wrapping correction to the $L = 4$ magnon energy from the twist-two spectrum. Now, one can immediately check that plugging $M = 1$ (recall that $S_{a_1,\ldots,a_n}(1) = \mathrm{sgn}(a_1)\cdots\mathrm{sgn}(a_n)$) into the wrapping correction (2.6.28) of Bajnok, Janik and Lukowski gives precisely the right hand side of (3.2.15).

We have thus verified that the $p = \pi$ magnon in $\mathcal{N} = 4$ Yang-Mills and the $\beta = 1/2$ physical magnon have the same energy up to four-loops, including wrapping effects, with the length of the spin chain such that wrapping starts at four loops, as is expected from the $\beta \to \beta + 1/2$ symmetry. We will now extend this check to longer spin chains. The formula (3.2.13) can be applied to extract the first wrapping correction to the $\beta = 1/2$ magnon energy for arbitrarily large L, but since the integrals $I_L^{(j+1)}$ are given explicitly up to $L = 9$ in [162] we will mainly limit ourselves to studying the first cases. We will however check that the transcendentality structures of the two magnon energies coincide also to higher L, and show that the coefficients of the maximal transcendentality terms match to all L.

For the $p = \pi$, undeformed case, we can obtain the wrapping correction for longer spin chains by evaluating equation (2.6.24). Performing the substitution $L \to L - 2$ to reflect that we are working with lengths as defined in the $SU(2)$ sector, and setting $M = 1$, we get

$$\gamma_{\mathrm{wrap}}^{(L)}(1) = -4^{L-2} g^{2L} (\gamma_1^{(L)}(1))^2 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{T_1(q,\,Q)^2}{R_1(q,\,Q)} \frac{1}{(q^2+Q^2)^{L-2}}\ , \tag{3.2.16}$$

with

$$R_1(q,\,Q) = P_1\left(\frac{1}{2}(q-i(Q-1))\right)P_1\left(\frac{1}{2}(q+i(Q-1))\right)\cdot$$
$$P_1\left(\frac{1}{2}(q+i(Q+1))\right)P_1\left(\frac{1}{2}(q-i(Q+1))\right)\ , \tag{3.2.17}$$

$$T_1(q,\,Q) = \sum_{j=0}^{Q-1}\left[\frac{1}{2j-iq-Q}+\frac{1}{2(j+1)-iq-Q}\right]P_1\left(\frac{1}{2}(q-i(Q-1))+ij\right)\ , \tag{3.2.18}$$

and

$$P_1(u) = 2i(u-u_1)\ , \tag{3.2.19}$$

where u_1 is the single root of the one-loop Baxter equation with $M=1$. As discussed earlier, for L even, the one-loop Baxter equation requires $u_1=0$, corresponding to precisely $p=\pi$. Evaluating the magnon dispersion relation (2.4.14) or (3.2.8), then gives directly the one-loop energy as $\gamma_1^{(L)}(1)=8$, for all L. Furthermore, we find

$$R_1(q,\,Q) = (q^2+(Q-1)^2)(q^2+(Q+1)^2)\ , \tag{3.2.20}$$

and

$$T_1(q,Q) = i\sum_{j=0}^{Q-1}\left[\frac{q-i(Q-1)+2ij}{2j-iq-Q}+\frac{q-i(Q-1)+2ij}{2(j+1)-iq-Q}\right] =$$
$$= -\sum_{j=0}^{Q-1}\left[2+\frac{i}{q-iQ+2ij}-\frac{i}{q-iQ+2i(j+1)}\right] \tag{3.2.21}$$

The last two terms in the summand cancel between successive values of j, giving

$$T_1(q,\,Q) = -2Q-\frac{i}{q-iQ}+\frac{i}{q+iQ} = -2Q\left(1-\frac{1}{q^2+Q^2}\right). \tag{3.2.22}$$

Introducing (3.2.20) and (3.2.22) into (3.2.16) for $M=1$, and using that, according to [94], the only pole that contributes to the integral after summing over Q is the one at $q=iQ$, gives our result

$$\gamma_{\rm wrap}^{(L)}(1) = -256\cdot 4^{L-2} i g^{2L}\sum_{Q=1}^{\infty}\mathrm{Res}_{q=iQ}\left[\frac{Q^2\left(1-1/(q^2+Q^2)\right)^2}{(q^2+(Q-1)^2)(q^2+(Q+1)^2)(q^2+Q^2)^{L-2}}\right]. \tag{3.2.23}$$

Evaluating this expression for $L=6$ and $L=8$ one has

$$\gamma_{\rm wrap}^{(6)}(1) = 128\,g^{12}\left[32\zeta(5)+28\zeta(7)-63\zeta(9)\right]\ , \tag{3.2.24}$$

$$\gamma_{\rm wrap}^{(8)}(1) = 768\,g^{16}\left[32\zeta(7)+64\zeta(9)+44\zeta(11)-143\zeta(13)\right]\ . \tag{3.2.25}$$

These expressions coincide with what is obtained from (3.2.13).

Going to even higher L requires more work, but one thing we can do with relative ease is to extract the coefficients of the maximum transcendentality parts of the magnon energies and show that they coincide. According to the expansion displayed in [162], up to at least $L = 9$ the ζ of maximum transcendentality is always present in the coefficients P_L, $I_L^{(1)}$ and $I_L^{(2)}$, but not in the $I_L^{(j)}$ for $j > 2$. Furthermore, applying the exact all-loop expression for $I_L^{(1)}$ and $I_L^{(2)}$, the highest transcendentality ζ enters in the same way in these integrals, but with opposite signs, making them cancel. This implies that the highest transcendentality ζ has coefficient $\frac{1}{2}P_L$. All in all, inserting the definition of P_L, we have that the maximum transcendentality part of the first wrapping correction at $\beta = 1/2$ is [3]

$$\delta\gamma^{1/2}_{\text{max. trans.}} = -64\, g^{2L} \binom{2L-3}{L-1} \zeta(2L-3) \; . \tag{3.2.26}$$

To evaluate the maximum transcendentality part of (3.2.23), let us calculate the relevant part of the residue by noting that the term of maximal transcendentality dominates in the limit $Q \to 0$. When expanding in terms of $(q - iQ)$, the expansions of the $1/(q^2 + (Q \pm 1)^2)$ factors will be subleading in this limit, and can simply be set to one (which is what they evaluate to at $q = iQ$, when $Q \to 0$). For example,

$$\frac{1}{q + i(Q-1)} = \frac{1}{i(2Q-1)} \sum_{j=0}^{\infty} i^j \left(\frac{q - iQ}{(2Q-1)}\right)^j \; , \tag{3.2.27}$$

for which the expansion coefficients do not receive an enhancement as $Q \to 0$, while the factor $1/(q + iQ)$, present in $1/(q^2 + Q^2)$, has the expansion

$$\frac{1}{q + iQ} = \frac{1}{2iQ} \sum_{j=0}^{\infty} i^j \left(\frac{q - iQ}{2Q}\right)^j \; . \tag{3.2.28}$$

The most singular term is then the one having the maximum number of $1/(q - iQ)$ terms, since these have to be compensated for by the enhanced $1/(q + iQ)$-terms. We can thus substitute the numerator $\left(1 - 1/(q^2 + Q^2)\right)^2$ for $1/(q^2 + Q^2)^2$. So the maximal transcendentality contribution is simply obtained by evaluating the residue

$$\mathrm{Res}_{q=iQ} \left[\frac{Q^2}{(q^2 + Q^2)^L}\right] \; , \tag{3.2.29}$$

which is equivalent to finding the $(L-1)-th$ expansion coefficient of $Q^2/(q + iQ)^L$. This, in turn is given by

$$\frac{Q^2}{(2iQ)^{2L-1}} \binom{-L}{L-1} = \frac{1}{4^{L-1}} \frac{1}{i} \frac{1}{Q^{2L-3}} \binom{2L-3}{L-1} \; . \tag{3.2.30}$$

[3] That the argument of the ζ of maximum transcendentality should be $2L - 3$ was conjectured in [167], for the $\mathcal{N} = 4$ theory.

where we have used that

$$\binom{-L}{L-1} = 2(-1)^{L-1}\binom{2L-3}{L-1} . \tag{3.2.31}$$

Substituting the residue of (3.2.23) for (3.2.30) and summing over Q immediately gives (3.2.26).

Summing up, there is a substantial amount of evidence that the energy of the magnon of momentum $p = \pi$ in the $\mathcal{N} = 4$ theory coincides with the anomalous dimension of the single impurity operator in the $SU(2)$ sector at $\beta = 1/2$ for all even spin chain lengths. This supports the conjecture that if one relaxes the cyclicity constraint, the spectrum of the $SU(2)$ spin chain is invariant under $\beta \to \beta + \frac{n}{L}$.

Before ending this section, let us also note that after the results presented in this section appeared, Beccaria and De Angelis [168] were able to rewrite (3.2.23) in the closed form

$$\frac{\gamma^{(L)}_{\text{wrap}}(1)}{g^{2L}} = -64\binom{2L-3}{L-1}\zeta(2L-3) + 128\sum_{l=1}^{\frac{L}{2}-1}\frac{2^{2l}l}{L-2l-1}\binom{2L-2l-3}{L-1}\zeta(2L-2l-3) . \tag{3.2.32}$$

This formula makes the transcendentality structure of the single-magnon wrapping correction clear, with terms proportional to the ζ-function evaluated at all odd values between $L-1$ and $2L-3$. In particular, it has the curious property of a vanishing rational part. The formula (3.2.32) is also necessary if one wants to, for example, expand the result around infinite L.

3.2.2 Consequences for the undeformed theory

The formula (3.2.32) provides an elegant answer to what the anomalous dimension is for the single-impurity operator is in the $SU(2)$ sector at $\beta = 1/2$, including the first wrapping correction for arbitrary L, but we can also ask ourselves what consequences the symmetry (3.2.1) has for the original $\mathcal{N} = 4$ theory.

To begin with, if we can calculate the anomalous dimension of the single-impurity operator at $\beta = 1/2$ for $L = 4$ to higher orders this will impose constraints on the spectrum of twist-two operators, through their analytical continuation to $M = 1$. In [13], this spectrum was calculated to five loops, but apart from the constraints imposed by BFKL, which the spectrum should satisfy when analytically continued to $M = -1$, one would like more evidence that supports this calculation. Since new elements enter at five-loops, such as the affecting of the wrapping correction by the dressing factor, one would like to have independent ways to verify it. It was therefore mentioned in [13] that the constraint at $M = 1$ stemming from the β-deformed theory could possibly be used, since at five loops it is technically easier to calculate the single-impurity operator dimension than any physical state in the $SL(2)$ sector.

But we can also take the constraints implied by (3.2.1) one step further. The symmetry implies that the anomalous dimensions of the single magnon operators, as given in (3.2.10), at $\beta = N/L$, coincide with the energy of the $p = 2\pi N/L$ magnon in the undeformed theory. For this set of momenta we thus have the wrapping correction

$$\begin{aligned}\delta E(p) = -2Lg^{2L}\Big[&\big(C_{L,0}\left(\frac{p}{2\pi}\right) - C_{L,L-1}\left(\frac{p}{2\pi}\right)\big)P_L \\ -2&\sum_{j=0}^{\lfloor \frac{L}{2} \rfloor -1}\big(C_{L,j}\left(\frac{p}{2\pi}\right) - C_{L,L-j-1}\left(\frac{p}{2\pi}\right)\big)I_L^{j+1}\Big]\,. \end{aligned} \tag{3.2.33}$$

Curiously, we obtain the correction for precisely those magnons that correspond to physical (periodic) states of the spin chain. One could object that magnons are only defined for asymptotically long spin chains, but what is meant by (3.2.33), is the eigenvalue of the dilatation operator acting on states of the form (2.3.11), which are perfectly well defined if one relaxes the cyclicity constraint. The equation (3.2.33) can thus be seen as a constraint on the dilatation operator itself, which is not known beyond three loops.

Another example of the importance that the β-deformed theory has on determining the structure of the $SU(2)$ sector in the undeformed theory is that of [169], where the properties of the Baxter Q-operator is studied for the $\mathcal{N} = 4$ theory by constructing it for the twisted theory and then taking the limit $\beta \to 0$.

3.3 New developments and outlook

Recently, a couple of papers advancing considerably the issue of wrapping corrections in the β-deformed theory have appeared. In [170], a proposal for how to introduce a β-twist into the Lüscher correction of the Konishi operator was given, which produces a result coinciding with the explicit calculation of [161]. In [171] a proposal for a Y-system for the β-deformed theory was put forth, conjectured to give the exact all-loop spectrum of the theory.

It would seem that, just like the case of the undeformed theory, a complete determination of the planar β-deformed spectrum in terms of an integrable model is not far away. There are still some issues left to understand, however. Firstly, it may be difficult to derive the symmetry (3.2.1) from the Y-system, since it applies to the entire algebra, where the cyclicity constraint is needed for consistency. The Y-system provides the eigenvalues of the dilatation operator, but new insights may be needed if one is to construct the form of the dilatation operator itself.

Secondly, it would be interesting to see if the Y-system could be derived from a Thermodynamic Bethe Ansatz, as it can in the original theory, since it is not obvious how to motivate the double Wick rotation underlying it if one has twisted boundary conditions in the spatial direction.

Chapter 4

Scattering amplitudes

This short chapter, is a pure review chapter, giving a quick overview over some recent developments in scattering amplitudes in $\mathcal{N} = 4$ SYM. The aim will be to note the existence of a hidden, dual superconformal symmetry, which together with the ordinary superconformal symmetry of the theory generates a Yangian symmetry, which in fact also appears in the case of the dilatation operator. This will provide the background for a conjecture relating the dual conformal symmetry to symmetries of the BFKL equation, describing color singlet exchange in the Regge limit of four-dimensional gauge theories, in chapter 5. For more extensive reviews on the subject, see [172].

4.1 Scattering amplitudes in $\mathcal{N} = 4$

Apart from phenomenological reasons, scattering amplitudes have always been an important tool for analyzing the properties of quantum field theories. However, $\mathcal{N} = 4$ Super Yang Mills is a conformal theory, and does therefore in a strict sense not have an S-matrix for particle scattering, due to the impossibility of defining asymptotic states. However, all gauge theory amplitudes will have infrared divergences, which must be regularized in some way, and in the presence of such an IR regulator, the conformal invariance is broken and scattering amplitudes can be defined as usual, even in this case.

And there are plenty of reasons for wanting to study these scattering amplitudes. Apart from a very fascinating structure, which we will describe briefly in this chapter, in AdS/CFT they provide a new means to check the correspondence. The reason is that if, for example, string theory implies that the amplitudes should have some non-trivial property, the correspondence implies that the same property should also be present at weak coupling.

We will denote an n-particle amplitude by $\mathcal{A}_n$, where the particles involved in the scattering are labeled by their momenta, $p_1, \ldots, p_n$, helicities $h_1, \ldots, h_n$, and color indices $a_1, \ldots a_n$(see figure 4.1). In $\mathcal{N} = 4$ all particles are massless, and thus have a well-defined helicity, which determines the particle type since the gluons have helicity ± 1, the fermions have $\pm\frac{1}{2}$, and the scalars have helicity 0. Also, they all transform in the adjoint representation of the gauge group, which we take to be $SU(N)$, implying that they have the same structure of color indices. Furthermore, we will be interested in on-shell amplitudes, in which $p_i^2 = 0$.

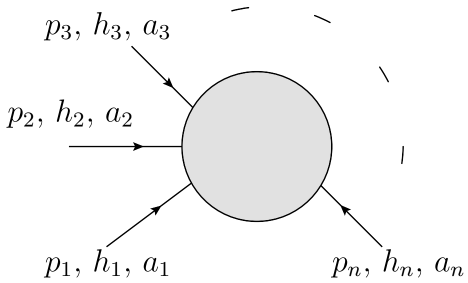

Figure 4.1: An n-particle scattering amplitude.

A given amplitude can be decomposed into terms proportional to products of traces of color matrices. For a simple explanation of how this works, see [173]. Once again we will take the planar limit, $N \to \infty$, in which one can neglect multiple traces. The amplitude can then be written (up to a numerical coefficient) as

$$\mathcal{A}_n \sim \delta^{(4)}\big(\sum_{i=1}^{n} p_i\big) \sum_{\sigma \in S_n/Z_n} g_{YM}^{n-2} \mathrm{Tr}\,[T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}]\, A_n^{h_{\sigma(1)},\ldots,h_{\sigma(n)}}(p_{\sigma(1)}, \ldots, p_{\sigma(n)}) \;, \quad (4.1.1)$$

where S_n is the group of permutations of n numbers, Z_n is the group of cyclic permutations and the T^a are color matrices. From now on we will focus on the residual amplitudes, of the form

$$A_n^{h_1,\ldots,h_n}(p_1, \ldots, p_n) \;, \quad (4.1.2)$$

which are called "color-ordered partial amplitudes", since we then do not have to worry about the color indices.

In the case of gluon scattering amplitudes, we will simply denote the helicities by $+$ and $-$. This type of amplitude is the most studied, and is especially important since they are identical at tree level for both QCD and $\mathcal{N} = 4$ (if all external particles are gluons the only way to introduce fermions or scalars would be via an internal loop). In fact, many of the rather curious properties of such amplitudes can be understood more directly from the maximally supersymmetric theory. For example, supersymmetry will imply that

$$A^{++\cdots+} = A^{-+\cdots+} = 0 \;, \quad (4.1.3)$$

all gluon amplitudes with no negative helicity gluons, or only one such gluon must vanish. The equality of QCD and $\mathcal{N}=4$ tree level gluon amplitudes then implies that this must be true for QCD as well. The difference between QCD and $\mathcal{N}=4$, is that in the former case (4.1.3) is only valid at tree level, while in the latter it holds to all orders.

The first configuration of helicities leading to a non-vanishing amplitude is that of two negative helicity gluons, giving what is known as a Maximally Helicity Violating amplitude (MHV). All 4 and 5 point amplitudes are MHV and in general such amplitudes turn out to have remarkable properties. To begin with, as discovered by Parke and Taylor [174], the tree-level gluon MHV amplitudes, despite being evaluated by summing over a large number of Feynman diagrams, are given by a very compact expression. In order to present this formula, we must introduce a notation, which is very powerful when treating on-shell scattering amplitudes, known as the spinor-helicity formalism (see for example [173]).

In this formalism we represent the on-shell momenta p_i^μ in terms of a pair of two-component spinors λ_α^i and $\tilde{\lambda}_{\dot{\alpha}}^i$. To arrive at this representation one uses that any four-vector p^μ can be represented by a bi-spinor $p_{\alpha\dot{\alpha}}$, given by

$$p_{\alpha\dot{\alpha}} = p^\mu \left(\sigma_\mu\right)_{\alpha\dot{\alpha}} \ , \quad \left(\sigma^\mu\right)_{\alpha\dot{\alpha}} = (1,\, \vec{\sigma})_{\alpha\dot{\alpha}} \ . \tag{4.1.4}$$

Using the anti-symmetric tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$, taking for example the convention

$$\varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{12} = -\varepsilon^{\dot{1}\dot{2}} = 1 \ , \tag{4.1.5}$$

to lower and raise indices, the square of the four-vector becomes

$$p^2 = p^\mu p_\mu = \frac{1}{2} p^{\dot{\alpha}\alpha} p_{\alpha\dot{\alpha}} \ . \tag{4.1.6}$$

So for on-shell momenta, with $p^2 = 0$, we will have $p^{\dot{\alpha}\alpha} p_{\alpha\dot{\alpha}} = 0$. This allows us to split the bi-spinor into a pair of two-component spinors

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \ , \tag{4.1.7}$$

since a single such spinor satisfies $\lambda_\alpha \lambda^\alpha = 0$.

Introducing the notation

$$\langle \psi \lambda \rangle \equiv \psi^\alpha \lambda_\alpha \ , \quad [\tilde{\psi} \tilde{\lambda}] \equiv \tilde{\psi}_{\dot{\alpha}} \tilde{\lambda}^{\dot{\alpha}} \tag{4.1.8}$$

for spinor contractions, together with the further abbreviation

$$\langle i\, j \rangle \equiv \langle \lambda^i\, \lambda^j \rangle \ , \tag{4.1.9}$$

the Parke-Taylor formula for the color-ordered partial MHV amplitude, where gluons j and k are the negative helicity ones, is given by

$$A_n = \frac{\langle j\, k \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n-1\, n \rangle} \ . \tag{4.1.10}$$

Curiously, this formula only depends on the λ spinors. But this is only a property of MHV amplitudes. When we have three negative helicity gluons, known as a Next to Maximally Helicity Violating amplitude (NMHV), the result also depends on the $\tilde{\lambda}$ spinors, as it will also for even more gluons of negative helicity (NNMHV, NNNHMV, etc). Still, the spinor-helicity variables give remarkably compact expressions for the scattering amplitudes, reflecting the simplifications obtained by a purely on-shell formalism.

Such simplicity is not at all obvious from an inspection of individual Feynman diagrams. Much progress was made with the discovery of procedures that constructs a tree-level amplitude recursively, in terms of amplitudes with fewer legs, such as the BCFW [175] and CSW [176] algorithms. Applying such an algorithm it is not difficult, for example, to show the Parke-Taylor formula (4.1.10). Still, if the amplitudes present such a high degree of structure, one suspects that there is some symmetry responsible for it.

Even though it is not the entire story, as we will see, an obvious symmetry that we have in this case is that of $\mathcal{N}=4$ supersymmetry. There is a very nice formalism for gathering together amplitudes which are related by supersymmetry into a single superamplitude. To do this one starts by collecting the different fields of the $\mathcal{N}=4$ supermultiplet into a super-wave function

$$\begin{aligned}\Phi(p,\eta) = G^{+}(p) + \eta^{A}\Gamma_{A}(p) + \frac{1}{2}\eta^{A}\eta^{B}S_{AB}(p) + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\varepsilon_{ABCD}\bar{\Gamma}^{D}(p) + \\ + \frac{1}{4!}\eta^{A}\eta^{B}\eta^{C}\eta^{D}\varepsilon_{ABCD}G^{-}(p)\ .\end{aligned} \tag{4.1.11}$$

Here, the η^A are four Grassmann variables, which are taken to have helicity $+\frac{1}{2}$, so that the entire wave-function has helicity $+1$. The supersymmetry generators are represented on the wave function (4.1.11) through

$$q^{A}_{\alpha} = \lambda_{\alpha}\eta^{A}\ , \quad \bar{q}_{A\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}}\frac{\partial}{\partial\eta^{A}}\ . \tag{4.1.12}$$

Superamplitudes are then given as functions of the momenta, or equivalently the spinors λ^i and $\tilde{\lambda}^j$, and the η^A_i, $\mathcal{A}_n(\lambda,\tilde{\lambda},\eta)$, with the understanding that a power expansion of the amplitude in terms of the η^A_i will give the different component amplitudes. For example, a term proportional to $\eta^1_1\eta^2_1\eta^3_1\eta^4_1$ will correspond to an amplitude having as its first particle a negative helicity gluon (the term having four η in (4.1.11)). Now, invariance under the generators $p_{\alpha\dot{\alpha}} = \sum_i p^i_{\alpha\dot{\alpha}}$ and $q^A_\alpha = \sum_i q^A_{i\,\alpha} = \sum_i \lambda^i_\alpha \eta^A_i$ requires that the superamplitudes take the form

$$\mathcal{A}_n(\lambda,\tilde{\lambda},\eta) = \delta^{(4)}(p_{\alpha\dot{\alpha}})\delta^{(8)}(q^A_\alpha)\mathcal{P}_n(\lambda,\tilde{\lambda},\eta)\ , \tag{4.1.13}$$

where $\delta^{(8)}(q^A_\alpha) = \prod_{A=1}^{4}\prod_{\alpha=1}^{2} Q^A_\alpha$ is a Grassmann delta function. We can now see directly why the gluon amplitudes with one or no negative helicity gluons vanish, since the expansion of the Grassmann delta gives the minimum number of η-factors as 8. In general, all the component amplitudes with 8 factors of η are called MHV amplitudes, even though they in general will not be pure gluon amplitudes. A general MHV amplitude must be proportional to the zeroth order (in the η-expansion) of $\mathcal{P}_n$, which, since it has no η-dependence, is the same for all particle configurations. Comparing with the Parke-Taylor amplitude (4.1.10), and noting that its numerator $\langle j\,k\rangle^4$ is precisely what is obtained by keeping the part of $\delta^{(8)}(\lambda_\alpha^i\eta_i^A)$ corresponding to two negative helicity gluons at positions j and k, one obtains directly the super-MHV amplitude

$$\mathcal{A}_n^{MHV}(\lambda,\,\tilde{\lambda},\,\eta) = \frac{\delta^{(8)}(\lambda_\alpha^i\eta_i^A)}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n-1\,n\rangle}\ , \tag{4.1.14}$$

first given by Nair [177].

Using this superformalism, together with some further developments, a construction of all tree level amplitudes in $\mathcal{N}=4$ has been carried out recently [178]. However, for a complete treatment of scattering amplitudes, and a chance to make contact with strong coupling results implied by AdS/CFT, it is clear that one must go beyond tree level.

It should not be surprising that this is easiest to do for MHV amplitudes. In fact, an arbitrary MHV (super)amplitude can be decomposed as

$$A_n^{MHV} = A_n^{MHV,\text{tree}} M_n\ , \tag{4.1.15}$$

where the tree-level factor $A_n^{MHV,\text{tree}}$ contains all the dependence on the particle types and helicities, while the loop correction M_n only depends on the momenta of the particles (and on any regulator one may introduce to regularize loop integrals). Among the MHV amplitudes, the simplest instance is the 4 point amplitude, which consequently has received much attention. For example, in [8, 179] it was calculated up to four loops using unitarity based methods.

In its simplest version, unitarity is used as follows: if we decompose the S-matrix as

$$S = 1 + iT\ , \tag{4.1.16}$$

where T is due to interactions, unitarity of the S-matrix $S^\dagger S = 1$ will imply

$$i(T^\dagger - T) = T^\dagger T\ . \tag{4.1.17}$$

The left hand side is twice the imaginary part of T, $2\mathrm{Im}T$. If we evaluate this expression at a given order g^{2L} in perturbation theory, it will express $\mathrm{Im}T$ at that order in terms of the right hand side, a sum of products of on-shell amplitudes at lower orders, which we can assume are known. In doing so, one can pictorially represent the right hand side by drawing a Feynman diagram at order g^{2L}, and then putting those propagators on-shell that correspond to intermediate states in the product of the two T-matrices. This leads to the Cutkosky cutting rules [180], and the propagators which have been put on-shell are called "cut" propagators. Having the imaginary part of T at order g^{2L}, one can then construct the real part by so-called dispersion integrals.

This method can then be generalized by allowing more cuts than the standard unitarity formula (4.1.17) allows [181]. Applying this generalized unitarity method many high loop results have been obtained. Returning to the case of the four-gluon amplitude, it was observed in [182] that it exhibits a curious iterative structure. This observation was extended to an all-loop conjecture in the paper [7] by Bern, Dixon and Smirnov, and is known as the BDS conjecture. Actually, based on evidence from the five point amplitude, and n-point amplitudes at one loop, the conjecture was stated to apply to all MHV amplitudes. Denoting the logarithm of the finite part of the loop correction M_n of (4.1.15) as $F_n \equiv \ln M_{n,\text{finite}}$, the BDS conjecture states that to all orders in perturbation theory

$$F_n = \frac{1}{4}\Gamma(g^2)F_n^{(1)} \ , \tag{4.1.18}$$

where $\Gamma(g^2)$ is the cusp anomalous dimension, given at the first perturbative orders by (2.5.16), and supposedly to all orders by the BES equation [87], and $F_n^{(1)}$, which is obtained by a one-loop calculation, is a function of the kinematical variables, but independent of the coupling. There is a general formula for $F_n^{(1)}$, which in the cases $n = 4$ and $n = 5$ gives

$$F_4^{(1)} = \frac{1}{2}\ln^2\frac{s}{t} + \text{const} \tag{4.1.19}$$

$$F_5^{(1)} = \frac{1}{2}\sum_{i=1}^{5}\ln\frac{s_{i,\,i+1}}{s_{i+1,\,i+2}}\ln\frac{s_{i+2,\,i+3}}{s_{i+3,\,i+4}} + \text{const} \ . \tag{4.1.20}$$

where s and t are the standard Mandelstam invariants, while $s_{i,\,i+1} = (p_i + p_j)^2$

This exponentiation of the finite part of the amplitude is similar to that of the IR divergent part, which has a structure that is well understood for all gauge theories, and in which the leading divergence is controlled by the cusp anomalous dimension. For the four and five point amplitudes, all available evidence is in favor of the BDS conjecture. And at four loops, the amplitude calculation of [8] produced for the first time a value for the four-loop cusp anomalous dimension, a value that was later reproduced in [87, 115] through the integrability based approach which was the topic of chapter 2.

However, starting from six points, and two loops the BDS conjecture was found to be no longer valid. This was first noticed in [183], where it was found that the BDS amplitude failed to present the correct analyticity properties in the high energy (multi-Regge) limit. It was only very recently that the remainder function $R_6^{(2)}$, the difference between the true two-loop six point amplitude and the BDS conjecture was calculated [184, 185], and shown to satisfy the high-energy constraints [186].

Once again, the fascinating structure implied by the BDS conjecture (and its failure starting from six points) can be understood from symmetry considerations. It turns out that the amplitudes exhibit a non-Lagrangian, hidden dual superconformal symmetry, different from the original superconformal symmetry, which is the topic of section 4.3. This dual conformal symmetry provides a constraint on the amplitudes for which the BDS conjecture is the minimal solution. For four and five points there is no additional liberty and the symmetry fixes the amplitudes uniquely, but starting from six points one has the freedom of adding dual superconformal invariants to the amplitude. The remainder function $R_6^{(2)}$ is thus a function of such dual superconformal invariants.

Furthermore, if the ordinary and dual superconformal symmetries are taken together, they generate a Yangian symmetry algebra [187], the topic of section 4.4, which is an infinite-dimensional symmetry algebra characteristic of integrable systems. This Yangian symmetry explains the structure present in tree level amplitudes to a great extent. The MHV amplitudes are determined uniquely, for example, explaining the Parke-Taylor formula (4.1.10) entirely in terms of symmetries. In general, however, it must be supplemented with certain analyticity conditions, such as the cancellation of certain non-physical "spurious" poles in order to fix the tree amplitudes [188].

Extending the Yangian symmetry to higher loops, however, is difficult, since the ordinary superconformal symmetry is broken starting from one-loop. One source of this breaking is the so-called collinear anomaly. At tree level this anomaly consists in the amplitudes containing non-superconformally invariant contact terms corresponding to configurations in which two or more particles become collinear. At lowest order this is not a serious problem since it only arrises for such degenerate kinematical configurations, but it will lead to a complete breaking of the symmetry at higher loops. For example, if we use the unitarity relation (4.1.17) to express a one-loop amplitude in terms of on-shell tree amplitudes, the integral over the intermediate on-shell momenta will certainly feel the collinear configurations and therefore be affected by the anomaly.

The solution to this problem seems to be to extend the concept of scattering amplitudes themselves. In [189], it was shown that the collinear anomaly could be cured at tree level by relating the collinear configurations to lower point amplitudes. This method was then extended to one loop in [190], with the result that the generators of the Yangian algebra could be deformed so that they annihilated the amplitudes at one loop as well. There still remains much to be understood in this approach, however, with the ultimate goal being an all-loop form for the representation of the Yangian.

A fascinating alternative development, also related to the Yangian symmetry, is the construction of a dual formulation of the S-matrix, in which all tree-level amplitudes, and the leading singularities of the amplitudes to all orders, are determined from a surprisingly compact Grassmannian integral [191]. One of the really interesting aspects of this formalism is that imposing Yangian symmetry basically fixes the integrand uniquely [193]. If the Grasmannian formula could be made rigorous, as indeed seems to be possible judging from the newer articles, this would mean that the Yangian determines the leading singularities of the amplitudes to all orders. In fact, very recently an extension of the BCFW recursion relations to loop amplitudes was given based on the Grasmannian picture [192], which would determine, in principle, any amplitude to any order.

Before ending this overview, let us comment that there has also been some exciting progress lately in the calculation of gluon scattering amplitudes at strong coupling. As we will also discuss briefly in section 4.3, it was shown by Alday and Maldacena in [194] that the problem of calculating and n-gluon scattering amplitudes to leading order at strong coupling is equivalent to calculating the area of a minimal surface in AdS_5, with a polygonal boundary with light-like sides given by the momenta of the particles involved in the scattering. This evaluation proved difficult for general n, but was finally solved in [195] by showing it to be equivalent to solving a Y-system of functional equations. Together with the presumed all-loop Yangian symmetry, the appearance of this Y-system suggests that, just as in the case of the spectrum of anomalous dimensions, an integrable model might underlie scattering amplitudes to all orders in $\mathcal{N} = 4$. At the moment, though, there is no obvious relation between this Y-system and that which appears for the anomalous dimensions. Curiously, scattering amplitudes seem much more tractable from the weak coupling side, in contrast with the anomalous dimension integrable model, which is basically constructed (at least in principle) from the strong coupling theory.

4.2 The superconformal symmetry algebra

Before moving on to discuss the dual superconformal symmetry in the next section, let us review the algebra and representation of the ordinary conformal symmetry on scattering amplitudes. Our reason for doing so is that the dual superconformal symmetry satisfies the same algebra, and that the truly interesting structure, the Yangian algebra, is obtained from the closure of the ordinary and dual algebras. The content of this section and the conventions used have been taken from [15].

To begin with, since we are working with a Lorentz invariant gauge theory, we have the Poincaré algebra, which in the spinorial notation consists of the Lorentz generators $\mathbb{M}_{\alpha\beta}$ and $\overline{\mathbb{M}}_{\dot\alpha\dot\beta}$, and the generator of translations $\mathbb{P}^{\alpha\dot\alpha}$. The Lorentz generators act in the canonical way on the corresponding type of indices:

$$[\mathbb{M}_{\alpha\beta},\, \mathbb{J}^{\gamma}] = \frac{1}{2}\delta^{\gamma}_{\beta}\,\mathbb{J}_{\alpha} + \frac{1}{2}\delta^{\gamma}_{\alpha}\,\mathbb{J}_{\beta}\ , \tag{4.2.1}$$

$$\left[\overline{\mathbb{M}}_{\dot\alpha\dot\beta},\, \mathbb{J}^{\dot\gamma}\right] = \frac{1}{2}\delta^{\dot\gamma}_{\dot\beta}\,\mathbb{J}_{\dot\alpha} + \frac{1}{2}\delta^{\dot\gamma}_{\dot\alpha}\,\mathbb{J}_{\dot\beta}\ , \tag{4.2.2}$$

together with the convention that spinorial indices are raised and lowered by the antisymmetric tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$, taking $\varepsilon_{12} = \varepsilon_{\dot 1\dot 2} = -\varepsilon_{12} = -\varepsilon_{\dot 1\dot 2} = 1$. In particular, this gives us the commutation relation for the momentum $\mathbb{P}_{\alpha\dot\alpha}$ with the Lorentz generators.

Next we have, of course, the $\mathcal{N} = 4$ supersymmetry algebra: introducing the fermionic generators $\mathbb{Q}_{\alpha A}$ and $\overline{\mathbb{Q}}^{A}_{\dot\alpha}$, where A is an $SU(4)$ R-symmetry index, taking the values $1, \ldots, 4$, it reads

$$\{\mathbb{Q}_{\alpha A},\, \overline{\mathbb{Q}}^{B}_{\dot\alpha}\} = \delta^{B}_{A}\mathbb{P}_{\alpha\dot\alpha}\ . \tag{4.2.3}$$

We also have the R-symmetry generator $\mathbb{R}^{A}{}_{B}$, which acts canonically on the R-symmetry indices.

Things start to get interesting when the conformal symmetry is added. Let y^{μ} denote space-time coordinates. To begin with, we have the dilatations, which simply expands the space-time according to

$$y^{\mu} \to \lambda y^{\mu}\ , \tag{4.2.4}$$

for any positive real number λ. The generator of infinitesimal dilatations is our old friend the dilatation operator $\mathbb{D}$, and its commutation with any other generator will give back the same generator multiplied by its classical dimension. For example, the classical dimension of $\mathbb{P}^{\alpha\dot\alpha}$ is one, so

$$[\mathbb{D},\, \mathbb{P}_{\alpha\dot\alpha}] = \mathbb{P}_{\alpha\dot\alpha}\ . \tag{4.2.5}$$

We then have the conformal inversions I, which act on the space-time coordinates as

$$y^{\mu} \to \frac{y^{\mu}}{y^{2}}\ . \tag{4.2.6}$$

These are finite transformations and can therefore not directly be included in the algebra, but if combine with $\mathbb{P}_{\alpha\dot\alpha}$, the generator of infinitesimal translations, then the string of transformations IPI will be infinitesimal, and correspond to a generator $\mathbb{K}_{\alpha\dot\alpha}$, known as the generator of special conformal transformations. Alternatively, instead of forming IPI we could have used the fermionic supersymmetry generators $\mathbb{Q}$ and $\overline{\mathbb{Q}}$, which produces the special superconformal generators $\overline{\mathbb{S}}_{A\,\dot\beta}$ and $\mathbb{S}^{A}_{\beta}$, through $I\mathbb{Q}I$ and $I\overline{\mathbb{Q}}I$, respectively.

The complete algebra for these generators is

$$
\begin{aligned}
\{\mathbb{Q}_{\alpha\,A},\,\overline{\mathbb{Q}}^{B}_{\dot\alpha}\} &= \delta^B_A\mathbb{P}_{\alpha\dot\alpha}\ , \qquad \{\mathbb{S}^A_\alpha,\,\overline{\mathbb{S}}_{\dot\alpha\,B}\} = \delta^A_B\mathbb{K}_{\alpha\dot\alpha}\ ,\\
[\mathbb{P}_{\alpha\dot\alpha},\,\mathbb{S}^A_\beta] &= \varepsilon_{\alpha\beta}\overline{\mathbb{Q}}^A_{\dot\alpha}\ , \qquad [\mathbb{K}_{\alpha\dot\alpha},\,\mathbb{Q}_{\beta\,A}] = \varepsilon_{\alpha\beta}\overline{\mathbb{S}}_{\dot\alpha\,A}\ ,\\
[\mathbb{P}_{\alpha\dot\alpha},\,\overline{\mathbb{S}}_{\dot\beta\ ,A}] &= \varepsilon_{\dot\alpha\dot\beta}\mathbb{Q}_{\alpha\,A}\ , \qquad [\mathbb{K}_{\alpha\dot\alpha},\,\overline{\mathbb{Q}}^A_{\dot\beta}] = \varepsilon_{\dot\alpha\dot\beta}\mathbb{S}^A_\alpha\ ,\\
[\mathbb{K}_{\alpha\dot\alpha},\,\mathbb{P}^{\beta\dot\beta}] &= \delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}\mathbb{D} + \mathbb{M}_\alpha{}^\beta\delta^{\dot\beta}_{\dot\alpha} + \overline{\mathbb{M}}_{\dot\alpha}{}^{\dot\beta}\delta^\beta_\alpha\ ,\\
\{\mathbb{Q}_{\alpha\,A},\,\mathbb{S}^B_\beta\} &= \varepsilon_{\alpha\beta}\mathbb{R}^B{}_A + \mathbb{M}_{\alpha\beta}\delta^B_A + \varepsilon_{\alpha\beta}\delta^B_A(\mathbb{D}+\mathbb{C})\ ,\\
\{\overline{\mathbb{Q}}^A_{\dot\alpha},\,\overline{\mathbb{S}}^B_{\dot\beta\,B}\} &= \varepsilon_{\dot\alpha\dot\beta}\mathbb{R}^A{}_B + \mathbb{M}_{\dot\alpha\dot\beta}\delta^A_B + \varepsilon_{\dot\alpha\dot\beta}\delta^A_B(\mathbb{D}-\mathbb{C})\ .
\end{aligned}
\tag{4.2.7}
$$

Here, $\mathbb{C}$ is a central charge that is actually not present in the case of the ordinary conformal symmetry, since the 'P' of $PSU(2,2|4)$ eliminates it, but we still leave it in the algebra since it appears for the dual conformal symmetry.

As for the representation of the generators on the amplitudes. We have already seen in the previous section the expressions

$$
p^{\alpha\dot\alpha} = \sum_i \lambda^\alpha_i\tilde\lambda^{\dot\alpha}_i\ , \quad q^{\alpha\,A} = \sum_i \lambda^\alpha_i\eta^A_i\ , \tag{4.2.8}
$$

for the generators of momentum and supersymmetry, respectively. Furthermore, we have the generator

$$
\bar{q}^{\dot\alpha}_A = \sum_i \tilde\lambda^{\dot\alpha}_i\frac{\partial}{\partial\eta^A_i} \tag{4.2.9}
$$

which together with (4.2.8) satisfies the SUSY algebra (4.2.3), and which has the correct representation on the super-wavefunction (4.1.11).

In [196], Witten constructed the representation for the special conformal transformations in the spinor-helicity formalism, and found it to be

$$
k_{\alpha\dot\alpha} = \sum_i \frac{\partial}{\lambda^\alpha_i}\frac{\partial}{\tilde\lambda^{\dot\alpha}_i}\ . \tag{4.2.10}
$$

Having the representations for $\mathbb{P}$, $\mathbb{Q}$, $\overline{\mathbb{Q}}$ and $\mathbb{K}$, the representations of all the other generators can be obtained from them by application of the algebra (4.2.7).

4.3 Dual conformal symmetry and the amplitude-Wilson loop duality

In this section we will discuss the hidden, dual superconformal symmetry, which together with the original superconformal symmetry generates a Yangian symmetry algebra, as we will discuss in section 4.4.

The dual conformal symmetry (without the fermionic generators) was first discovered as a formal property of the loop integrals appearing in the perturbative expansion of MHV gluon amplitudes, and whose classification had been initiated in [182]. In [197] it was noted that If one introduces a set of variables x_i, related to the external, incoming particle momenta p_i, $i = 1 \ldots n$, through

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu \equiv x_{i\,i+1}^\mu \ , \tag{4.3.1}$$

the integrals are formally covariant under a conformal group acting on the x variables in the same way as the ordinary superconformal symmetry acts on spatial coordinates. Due to the difference involved in the definition (4.3.1), the amplitudes are obviously invariant under translations of the x, and they are also invariant under Lorentz transformations, since they are so under Lorentz transformations of the p^μ. The new statement is that the loop integrals are formally invariant under the conformal inversions

$$x_i^\mu \to \frac{x_i^\mu}{x_i^2} \ . \tag{4.3.2}$$

The simplest example of a loop integral is the one-loop scalar box diagram, represented pictorially in figure 4.2. In $\mathcal{N} = 4$ SYM, the one-loop correction to the four-particle amplitude is simply proportional to this box diagram [198], which is a special property of maximally supersymmetric theories [1].

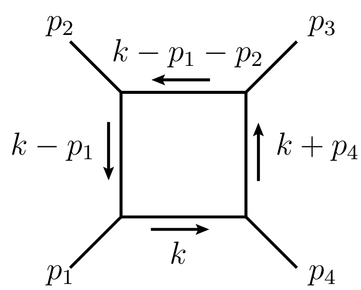

Figure 4.2: The scalar box diagram, giving the one-loop correction to four particle scattering amplitude in $\mathcal{N} = 4$ Super Yang Mills, in terms of incoming momenta p_i, and an integral over the loop momentum k.

The integral which one obtains when evaluating this diagram is

$$\int \frac{d^4k\,(p_1+p_2)^2(p_3+p_4)^2}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} \ , \tag{4.3.3}$$

[1]By Veltman-Passarino reduction [199] a general one-loop four-point amplitude can be represented in terms of a scalar box, triangle and bubble integral.

which becomes, after introducing x-variables (4.3.1), and defining a new integration variable x_I through $k = x_{1I}$,

$$\int \frac{d^4x_I \, x_{13}^2 x_{24}^2}{x_{1I}^2 x_{2I}^2 x_{3I}^2 x_{4I}^2} \ . \tag{4.3.4}$$

This expression is formally invariant under (4.3.2) as is easily seen since the squared differences $\boldsymbol{x}_{ij}^2$, and the measure d^4x_I transform as

$$x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2} \ , \ \text{ and } \ d^4x_I \to \frac{d^4x_I}{x_I^8} \ , \tag{4.3.5}$$

respectively.

The covariance is only formal, since if the external particles are on-shell, with $p_i^2 = 0$, this integral diverges [2], and so the change of integration variable implied by (4.3.5) is not, strictly speaking, allowed. Still, the dual conformal symmetry seemed to mean something since all the integrals found in the perturbative expansion satisfies this formal symmetry. Curiously, in [8], the four-loop four-point amplitude was analyzed, and it was noted that of the 10 dual conformal integrals that the authors could write down, only 8 actually contributed to the amplitude. A similar picture emerges at five loops, where the authors of [200] identified 59 dual conformally invariant integrals, of which only 34 contributed to the amplitude. In [201] a possible explanation for this selection was given. It turns out that if one attempts to regularize the integrals by keeping the external momenta off-shell, $p_i^2 \neq 0$, the integrals which give finite answers are precisely those which contribute to the amplitudes. The off-shell approach is not completely rigorous, since one cannot guarantee that the off-shell amplitudes do not have non-dual conformally invariant terms which vanish on-shell, but the same conclusion can be reached with a more recent approach, based on a Higgs-inspired regularization [202]. In this case, the different particles acquire independent masses m_i, which transform non-trivially and independently under dual conformal transformations.

A step towards understanding the origin of the dual conformal symmetry came in [194], where Alday and Maldacena identified the strong coupling dual, via AdS/CFT, of the gluon scattering amplitude. In order to calculate the scattering amplitude one must evaluate the area of a minimal surface in AdS_5, whose boundary, lying on the four-dimensional boundary of the AdS-space identified as where the $\mathcal{N} = 4$ theory lives, takes the form of a polygon, with sides given by the light-like momenta of the particles involved in the scattering process. Curiously, the evaluation of the area of the four-gluon minimal surface could be interpreted as the expectation value of a Wilson loop, defined by light-like segments having cusps at precisely the points x_i, given by (4.3.1). The Wilson loop

[2] This type of integral had already appeared in the study of four-point functions, in which case the x are given by actual space-time points for operator insertions, and the integrals are finite (as long as the x_{ij} are not light-like). This implies that the integrals must be given by conformal invariants.

closes since momentum conservation requires that $x_{n+1} = x_1$. Furthermore, the change of variable (4.3.1) had the interpretation of a T-duality. The dual conformal symmetry can then simply be interpreted as an ordinary conformal symmetry of the space that the Wilson loop lives in.

Evidence for this duality between scattering amplitudes and Wilson loops with light-like cusps, stating that the finite parts of the two are equal, was then produced at weak coupling, in [201, 203] at one loop, in [204] at two loops and four points and in [205] at two loops and five points.

The origin of the dual conformal symmetry was then made fully clear in [207], where it was found that the dual and ordinary conformal symmetries of the string world-sheet could be interchanged by performing a sequence of bosonic and fermionic T-dualities. The latter are based on fermionic isometries stemming from supersymmetry transformations, and behave in some ways like bosonic T-dualities based on non-compact isometries. Such T-dualities will in general not be symmetries of the full string theory, but instead of the theory obtained by restricting to the lowest order of the genus expansion, corresponding precisely to taking the planar limit. From this analysis we would therefore only expect dual conformal symmetry in the planar limit. This can also be understood by noting that without planarity there is no natural ordering of the momenta p_i, and therefore no natural way to introduce the dual variables x_i.

So what consequences does the dual conformal symmetry have? To begin with, the four and five point amplitudes are uniquely fixed to all loops! In [201] this was explained to be the case for the four point amplitude, where the assumption of dual conformal symmetry of the off-shell amplitude leads to a relation between the divergent and finite parts of the amplitude when going on-shell. And since there are no dual conformal invariants for four points x_i in which $x_{i\,i+1}^2 = 0$ this determines the amplitude uniquely. This also explains why the cusp anomalous dimension appears in the finite part of the amplitude.

However, as mentioned above the off-shell treatment is not entirely rigorous, and it furthermore is not known what its strong-coupling equivalent would be, making it difficult to justify its dual conformal invariance through a fermionic T-duality. But in [205] the issue was studied in dimensional regularization. In this case the regulator breaks the dual conformal symmetry, but in a controlled way. This allowed for the derivation of an anomalous ward identity, taking the form

$$K^\mu F_n \equiv \sum_{i=1}^{n} \left(2x_i^\mu x_i^\nu \frac{\partial}{\partial x_{i\,\nu}} - x_i^2 \frac{\partial}{\partial x_{i\,\mu}} \right) F_n = \frac{1}{4} \Gamma_{\text{cusp}}(g^2) \sum_{i=1}^{n} \log \frac{x_{i\,i+2}^2}{x_{i-1\,i+1}^2} x_{i\,i+1}^\mu \ , \tag{4.3.6}$$

where K^μ is the generator of dual special conformal transformations, F_n is the logarithm of the finite part of M_n, the loop correction to the MHV amplitudes, as defined in (4.1.15), and $\Gamma_{\text{cusp}}(g^2)$ is the cusp anomalous dimension.

Interestingly, the BDS conjecture (4.1.18) is a solution to (4.3.6) for all n. And for light-like segments, satisfying $x_{i\,i+1}^2 = 0$, there are no dual conformal invariants for $n = 4$ and $n = 5$, implying that the four and five point amplitudes are uniquely constrained to be given by the BDS formula! Assuming dual conformal invariance (as implied through AdS/CFT and the fermionic T-duality) this thus proves the BDS conjecture for four and five point amplitudes. For $n \geq 6$ the story is different due to the appearance of dual conformal invariants. In the case of the six point amplitude we have the invariant cross ratios

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \ , \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \ , \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2} \ , \tag{4.3.7}$$

and dual conformal symmetry admits any function of them. And as mentioned in the previous section, the difference between the true amplitude and the BDS conjecture did indeed consist of a non-zero remainder function. Since the remainder function is not fixed by the dual conformal symmetry, there is always the possibility that the duality with Wilson loops also fails in this case. However, fortunately this turns out not to be the case [206].

So far, the entire discussion has centered around loop integrals of MHV amplitudes. However, from the fermionic T-duality of the string world-sheet one would expect amplitudes to be dual superconformally invariant in general. To be able to show this one must understand how the fermionic generators act on amplitudes, and also how the symmetry is represented on tree amplitudes. In [15], where tree-level dual superconformal symmetry was studied systematically, it was found that the two issues must be dealt with simultaneously since in general dual conformal transformations will mix components of the super-wavefunctions related by supersymmetry.

Given that the spinor-helicity variables are more natural from the point of view of the tree amplitudes than the x-variables, the first thing one must do is to lift the action of the dual conformal inversions to the λ variables. In terms of spinor-helicity variables, the definition (4.3.1), takes the form

$$x_{i\,i+1}^{\dot\alpha\alpha} = \tilde\lambda_i^{\dot\alpha} \lambda_i^{\alpha} \ , \tag{4.3.8}$$

providing the constraint

$$x_{i\,i+1}^{\dot\alpha\alpha} \lambda_{i\,\alpha} = 0 \ . \tag{4.3.9}$$

Requiring this constraint to be invariant under dual conformal inversions I will then imply that the λ must transform as

$$I[\lambda_i^{\alpha}] = \kappa_i x_i^{\dot\alpha\beta} \lambda_{i\,\beta} \ , \tag{4.3.10}$$

where κ_i is an arbitrary x-dependent factor. The transformation of the $\tilde\lambda$ is then fixed since $\tilde\lambda$ is determined from the x and λ, due to (4.3.8), as

$$\tilde\lambda_i^{\dot\alpha} = \frac{x_{i\,i+1}^{\dot\alpha\alpha} \lambda_{i+1\,\alpha}}{\langle i\, i+1\rangle} \tag{4.3.11}$$

which one can show implies that

$$I[\tilde{\lambda}_i^{\dot{\alpha}}] = \frac{1}{\kappa_i x_i^2 x_{i+1}^2} \lambda_{i\,\dot{\beta}} x_i^{\dot{\beta}\alpha} \ . \tag{4.3.12}$$

The dual conformal invariance of the amplitudes should not depend on the choice of κ_i, but some choices will simplify the expressions. In particular, if one chooses $\kappa_i = 1/x_i^2$, one finds the simple transformation rule

$$I\left[\langle i\, i+1\rangle\right] = \left(x_i^2\right)^{-1} \langle i\, i+1\rangle \ , \quad I\left[\,[i\, i+1]\,\right] = \left(x_{i+2}^2\right)^{-1} [i\, i+1] \tag{4.3.13}$$

for spinor contractions of spinors corresponding to adjacent particles. For particles which are not adjacent there is no such simple transformation rule.

Using these transformations (and taking into consideration the transformation of the momentum-conserving delta function) one can show the covariance of any tree-level gluon amplitude having a so-called split-helicity configuration. These are amplitudes in which all the gluons of negative helicity are adjacent, without any positive helicity gluons in between. However, for more general helicity configurations the amplitude will fail to respect dual conformal invariance, as they will mix among themselves under such transformations. The solution is to introduce the formalism of the super-wavefunctions, where the factors of η determine the helicities of the component wavefunctions, and let the dual conformal transformations act on the η variables as well.

A further step that must be taken is to introduce a new set of "dual" fermionic variables $\theta_i^{A\,\alpha}$. The change of variables (4.3.1) automatically imposes momentum conservation through the differences of the x-variables, provided they satisfy the cyclicity $x_{n+1} = x_1$, and we can analogously introduce such a set of fermionic variables related to supersymmetry. We saw in the previous section that the annihilation of the amplitudes by the q_α^A imposed a Grassmann delta function. We can write this condition as

$$\sum_{i=1}^{n} \lambda_i^\alpha \eta_i^A = 0 \ , \tag{4.3.14}$$

which allows us to perform the change of variables

$$\theta_i^{A\,\alpha} - \theta_{i+1}^{A\,\alpha} = \lambda_i^\alpha \eta_i^A \ , \tag{4.3.15}$$

with $\theta_{n+1} = \theta_1$ then automatically imposing ordinary supersymmetry. If we want, we can then use (4.3.15) to express the η as

$$\eta_i^A = \frac{(\theta_{i\,i+1})^{A\,\alpha} \lambda_{i+1\alpha}}{\langle i\, i+1\rangle} \ . \tag{4.3.16}$$

Having the dual fermionic variables, the dual supersymmetry generators $Q_{A\,\alpha}$ are then simply given by

$$Q_{A\,\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial \theta_i^{A\,\alpha}} \, . \tag{4.3.17}$$

Then, since the dual conformal translations are generated by

$$P_{\alpha\dot{\alpha}} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}} \tag{4.3.18}$$

the $\mathcal{N}=4$ supersymmetry algebra

$$\{Q_{A\alpha}, \overline{Q}^B_{\dot{\alpha}}\} = \delta_A^B P_{\alpha\dot{\alpha}} \tag{4.3.19}$$

is satisfied by setting $\bar{Q}^A_{\dot{\alpha}} = \sum_{i=1}^{n} \theta_i^{A\alpha} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}}$. However, acting with this expression on the right hand side of (4.3.11) produces the right-hand side of (4.3.16), so that in order for the generator to act correctly on the $\tilde{\lambda}$-variables, we must add an additional term, with the result that

$$\overline{Q}^B_{\dot{\alpha}} = \sum_{i=1}^{n} \left[\theta_i^{A\alpha} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}} + \eta_i^A \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\alpha}}} \right] \, . \tag{4.3.20}$$

Since the θ play the role of fermionic coordinates in the dual superspace, where the x are the bosonic coordinates, it is a standard result that

$$I\left[\theta_i^{A\alpha}\right] = \left(x_i^{-1}\right)^{\dot{\alpha}\beta} \theta^A_{i\,\beta} \, . \tag{4.3.21}$$

Using (4.3.16) one can then determine the properties of the η_i^A under dual conformal inversions. The final form for the generator K^μ of dual special conformal transformations, defined as usual through an ITI transformation, where T is an infinitesimal translation, is then obtained as

$$K^{\dot{\alpha}\alpha} = \sum_{i=1}^{n} \left[x_i^{\dot{\beta}\alpha} x_i^{\dot{\alpha}\beta} \frac{\partial}{\partial x_i^{\dot{\beta}\beta}} + x_i^{\dot{\alpha}\beta} \theta_i^{B\,\alpha} \frac{\partial}{\partial \theta_i^{B\,\beta}} + x_i^{\dot{\alpha}\beta} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\dot{\beta}\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} + \tilde{\lambda}_i^{\dot{\alpha}} \theta_{i+1}^{B\,\alpha} \frac{\partial}{\partial \eta_i^B} \right] \, . \tag{4.3.22}$$

If some of the variables have been eliminated due to constraints such as (4.3.11) or (4.3.16), all we have to do is discard the corresponding derivatives to obtain the generators for the reduced set of variables.

The representation for the rest of the generators of the dual superconformal algebra can then be obtained, as in the case of the ordinary superconformal case, simply by (anti)-commuting $Q_{A\,\alpha}$, $\overline{Q}^B_{\dot{a}}$, $P_{\alpha\dot{a}}$ and $K^{\dot{\alpha}\alpha}$.

The generators $Q_{A\,\alpha}$ and $P_{\alpha\dot{a}}$ are automatically satisfied for the amplitudes due to the definition of the dual variables, so the important generator to check is that of special conformal transformations $K^{\dot{\alpha}\alpha}$. That all tree amplitudes indeed had this symmetry was established in [208].

A property of the dual conformal symmetry is that not all generators annihilate the amplitude. Instead, some, such as the generator for special conformal transformations, correspond to covariances instead on invariances. As a consequence, one has, for example, for the dual dilatation operator that $D\mathcal{A}_n = -n\mathcal{A}_n$. Furthermore, it turns out that the dual generators $\overline{Q}$ and $\overline{S}$ coincide with the generators $\overline{s}$ and $\overline{q}$ of the original algebra, respectively, and therefore annihilate the algebra. In order for the anti-commutation relation in (4.2.7) involving $\overline{Q}$ and $\overline{S}$ to hold, given that the Lorentz generators also annihilate the algebra, the dual superconformal algebra requires the existence of a central charge C, acting on amplitudes just like the dilatation operator, as $C\mathcal{A}_n = -n\mathcal{A}_n$. This implies that the dual superconformal symmetry algebra is in fact not $PSU(2,2|4)$, but $SU(2,2|4)$.

4.4 Yangian symmetry

Knowing that tree level amplitudes have both ordinary and dual superconformal symmetry algebras, a natural question is what is the closure of the two algebras, i.e. what they generate when taken together. This question was studied in [187] by Drummond, Henn and Plefka, with the result that an infinite-dimensional Yangian symmetry is obtained. In this section we will review some of the steps that they took since we will perform a closely related analysis in the context of BFKL in chapter 5.

For bosonic symmetry algebras, a Yangian [209] is generated by a set of elements $J_a^{(0)}$ and $J_a^{(1)}$, which we can call level 0 and level 1 respectively, that satisfy

$$[J_a^{(0)}, J_b^{(0)}] = f_{ab}{}^c J_c^{(0)} \ , \tag{4.4.1}$$

and

$$[J_a^{(1)}, J_b^{(0)}] = f_{ab}{}^c J_c^{(1)} \ , \tag{4.4.2}$$

for some structure constants $f_{ab}{}^c$, as well as the Serre relations

$$\begin{aligned}
&[J_a^{(1)}, [J_b^{(1)}, J_c^{(0)}]] + [J_b^{(1)}, [J_c^{(1)}, J_a^{(0)}]] + [J_c^{(1)}, [J_a^{(1)}, J_b^{(0)}]] = \\
&\qquad h f_{ak}{}^d f_{bl}{}^e f_{cm}{}^f f^{klm} \{J_d^{(0)}, J_e^{(0)}, J_f^{(0)}\} \ , & (4.4.3)\\
&[[J_a^{(1)}, J_b^{(1)}], [J_c^{(0)}, J_d^{(1)}]] + [[J_c^{(1)}, J_d^{(1)}], [J_a^{(0)}, J_b^{(1)}]] = \\
&\qquad h \left(f_{al}{}^g f_{bm}{}^e f_{kn}{}^f f^{lmn} f_{cd}{}^k + f_{cl}{}^g f_{dm}{}^e f_{kn}{}^f f^{lmn} f_{ab}{}^k \right) \{J_g^{(0)}, J_e^{(0)}, J_f^{(0)}\} & (4.4.4)
\end{aligned}$$

where h is a number which depends on conventions, and where $\{\cdot, \cdot, \cdot\}$ is the symmetrized triple product. Satisfying (4.4.1)-(4.4.4) guarantees that an infinite dimensional algebra is obtained. A key point in this construction is that one cannot judge from a Lie-algebra whether the Serre relations are satisfied (other than in special cases, such as $SU(2)$, where the first relation (4.4.3) becomes $0 = 0$) and one must look at a particular representation.

In $\mathcal{N} = 4$, we also have fermionic generators requiring the commutators to be exchanged for anti-commutators when appropriate, as well as the introduction of some signs in these equations. In particular, the first Serre relation (4.4.3) becomes [187]

$$[J_a^{(1)}, [J_b^{(1)}, J_c^{(0)}\}\} + (-1)^{|a|(|b|+|c|)}[J_b^{(1)}, [J_c^{(1)}, J_a^{(0)}\}\} + (-1)^{|c|(|a|+|b|)}[J_c^{(1)}, [J_a^{(1)}, J_b^{(0)}\}\} =$$
$$h(-1)^{|r||m|+|t||n|} f_{ak}{}^{d} f_{bl}{}^{e} f_{cm}{}^{f} f^{klm} \{J_d^{(0)}, J_e^{(0)}, J_f^{(0)}] \ , \tag{4.4.5}$$

where $|i|$ is the degree of the generator J_i.

A simple choice for the level zero generators $J_a^{(0)}$ is the ordinary superconformal algebra. Furthermore, with this choice we do not have to worry about the second Serre relation, since, as discussed in [211], it is implied by the first relation for most semi-simple (super)-Lie algebras, including $PSU(2, 2|4)$.

One would then like to extract a set of new generators from the dual superconformal algebra which could be identified with the level 1 generators. These new generators are then required to satisfy (4.4.2) and (4.4.5). The Serre relations are rather tedious to check, but fortunately there is a short-cut: if we decompose the level zero generators in terms of representations acting on the individual particles as

$$J_a^{(0)} = \sum_i^n J_{ia}^{(0)} \ , \tag{4.4.6}$$

as we saw could be done in section 4.2, and define a set of level 1 generators by the bi-local form

$$J_a^{(1)} = f_a{}^{bc} \sum_{1 \le j < i \le n} J_{jb}^{(0)} J_{ic}^{(0)} \ , \tag{4.4.7}$$

where the indices of the structure constants have been raised by the metric of the algebra, then the commutation relation (4.4.2) is guaranteed, as can easily be checked by introducing (4.4.7) in (4.4.2) and using the Jacobi identity. Furthermore, in [211] it was explained that the Serre relation is automatically satisfied, using (4.4.7), if the adjoint representation appears only once in the tensor product of the single particle representation with its conjugate. And this is indeed the case for the on-shell gluon multiplet. So if the dual superconformal symmetry can be shown to imply that the level one generators as defined by the bi-local form annihilate the amplitude, the Yangian symmetry will follow.

To show this there are a series of technical steps that must be taken. The first obstacle to combining the superconformal symmetries is that the dual conformal symmetry is most naturally written in terms of the x and θ variables, but their introduction will automatically impose momentum conservation and supersymmetry through (4.3.1) and (4.3.15) due to the cyclic identifiactions $x_{n+1} = x_1$ and $\theta_{n+1} = \theta_1$, making it impossible to have the ordinary momentum and supersymmetry generators alongside the dual conformal algebra. A preliminary step that must be done is therefore to relax this cyclicity condition, allowing for $n+1$ independent x and θ-variables, and instead include delta-functions $\delta^{(4)}(x_{n+1} - x_1)\delta^{(8)}(\theta_1 - \theta_{n+1})$ in the amplitudes themselves to impose momentum conservation and supersymmetry.

Next, one must adjust the dual generators so that they annihilate the amplitudes, as several of them generate covariances rather than invariances. It was shown in [15] that the momentum and SUSY-preserving delta-function pre-factor of the amplitudes is a dual superconformal invariant. For MHV amplitudes, the tree-level amplitude is given by (4.1.14), and its properties under dual conformal transformations stem entirely from the denominator $\langle 1\,2\rangle \cdots \langle n-1\,n\rangle$. Applying (4.3.22), and the corresponding formula for the superconformal generator S^A_α, one can check that

$$K^{\alpha\dot{\alpha}}\mathcal{A}_n = -\sum_{i=1}^{n} x_i^{\alpha\dot{\alpha}}\mathcal{A}_n\ , \quad S^A_\alpha \mathcal{A}_n = -\sum_{i=1}^{n}\theta^A_{i\alpha}\mathcal{A}_n \tag{4.4.8}$$

so that the amplitudes are annihilated by

$$\tilde{K}^{\alpha\dot{\alpha}} = K^{\alpha\dot{\alpha}} + \sum_{i=1}^{n} x_i^{\alpha\dot{\alpha}} \quad \text{and} \quad \tilde{S}^A_\alpha = S^A_\alpha + \sum_{i=1}^{n}\theta^A_{i\alpha}\ . \tag{4.4.9}$$

These are in fact the only new generators that are obtained from the dual conformal symmetry since the other dual generators are directly related to generators of the original superconformal algebra. For example, the dilatation generators are related through

$$D = d - n\ , \tag{4.4.10}$$

where, as usual, we use lowercase letters to denote the original algebra and uppercase letters for the dual algebra.

Finally, one must write the original and dual symmetries in a common language. From their construction, we know how dual generators such as (4.3.22) act on functions of the variables λ, x, η and θ, but it is not obvious how the generators of the original algebra act on the dual variables. A solution to this problem is to rewrite the dual generators entirely in terms of the λ and η variables. To do so one applies the definition (4.3.8) and (4.3.15) and solves for x_i and θ_i, giving

$$x_i^{\alpha\dot{\alpha}} = x_1^{\alpha\dot{\alpha}} - \sum_{j=1}^{i-1}\lambda^\alpha_j\tilde{\lambda}^{\dot{\alpha}}_j\ , \quad \theta_i^{\alpha A} = \theta_1^{\alpha A} - \sum_{j=1}^{i-1}\lambda^\alpha_j\eta^A_j\ , \tag{4.4.11}$$

while x_1 and θ_1 are left arbitrary. When performing this change of variable on $\tilde{K}$ and $\tilde{S}$, all the x_1 and θ_1-dependence either cancels, or multiplies generators of the original superconformal algebra. In the end, one obtains the new generators K' and S', written entirely in terms of the spinor-helicity variables. For example,

$$K' = -\sum_{i=1}^{n}\left[\sum_{j=1}^{i-1}\lambda_j^{\beta}\tilde{\lambda}_j^{\dot{\alpha}}\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\beta}} + \sum_{j=1}^{i}\lambda_j^{\alpha}\tilde{\lambda}_j^{\dot{\beta}}\tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\beta}}} + \sum_{j=1}^{i}\lambda_j^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}\eta_j^{B}\frac{\partial}{\partial\eta_i^{B}} + \sum_{j=1}^{i-1}\lambda_j^{\alpha}\tilde{\lambda}_j^{\dot{\alpha}}\right]. \tag{4.4.12}$$

As a consequence of the change of variable, the generator is no longer written as a sum over single particle representations. This is good news, however, since this is precisely the type of structure that appears in the bi-local representation (4.4.7). And indeed, the authors of [187] show that, after subtracting some further terms involving only generators of the original algebra, S' and K' can be put in the bi-local form. From (4.4.2) one then obtains the rest of the level one generators. It therefore follows that all tree-level amplitudes of $\mathcal{N}=4$ Super Yang Mills are invariant under a Yangian algebra.

The construction that we have just outlined may seem a bit asymmetric since it treats the original and dual conformal symmetries differently. Since the two symmetry algebras are mapped into each other under fermionic T-duality [207], there should exist a T-dual version of this construction in which the dual conformal generators constitute the level 0 algebra and one can use the ordinary conformal generators to construct the level one algebra. This is indeed the case, and this construction was performed in [210].

The appearance of the Yangian algebra is an exciting development since Yangian algebras are normally characteristic of integrable models. Also, the same Yangian appears as a symmetry of the dilatation operator [96, 211, 97]. It could thus be the case that some integrable model underlies the scattering amplitudes of $\mathcal{N}=4$. Before such a model can be identified, one must, however, extend the Yangian symmetry to higher orders, as we discussed in section 4.1. Given the non-triviality of this problem, its solution is bound to provide many insights into the nature of scattering amplitudes.

Chapter 5

The BFKL equation, integrability and a dual conformal symmetry

We will now consider a particular high-energy limit of scattering amplitudes known as the Regge limit, focusing to a great extent on the BFKL equation giving the amplitude for color-singlet exchange. An interesting property of the Regge limit is that basically only the gluons will contribute at leading order implying that the same results are obtained for QCD and $\mathcal{N} = 4$ Super Yang Mills. Taking the integrability of $\mathcal{N} = 4$ as a point of departure, this can explain the large amount of symmetry, which is rather mysterious from the point of view of QCD, found in this limit.

In section 5.1 we will introduce the Regge limit, and try to motivate the BFKL equation. In section 5.2 we will then present briefly an integrable model appearing in this limit, which motivates a study, described in section 5.3, of whether a remnant of the symmetries of $\mathcal{N} = 4$ Super Yang Mills can be identified in the BFKL framework. In the process a new symmetry of the BFKL equation is identified, which is interesting by itself, regardless of potential links to $\mathcal{N} = 4$ and integrability.

Here, we will not present the literature in a comprehensive way, nor present an overview over the current state of the field, but rather limit ourselves to introducing those concepts that are necessary for a understanding of the results of section 5.3.

5.1 The Regge limit and the BFKL equation

In this section we will briefly review the Regge limit of QCD and the Balitsky, Fadin, Kuraev, Lipatov (BFKL) [10] equation. As a preliminary step we will introduce the reggeized gluon, which is a key ingredient in this context, and gives the leading behaviour in the high energy limit of certain scattering amplitudes. We will then present leading logarithmic BFKL and the pomeron. For a more detailed account, see the review [213], or the excellent textbook [212] by Forshaw and Ross, which also includes an overview of the experimental motivations for the Regge limit and BFKL.

5.1.1 The Regge Limit

Consider a 4 point scattering amplitude $\mathcal{A}(s, t)$, parametrized by Mandelstam variables $s \equiv (p_1 + p_2)^2$, and $t \equiv (p_2 + p_3)^3$. Suppose that we are also in the "physical" kinematical regime in which $s > 0$ and $t < 0$, and in which we interpret particles 1 and 2 as "incoming", while particles 3 and 4 are "outgoing". The Regge limit [9] then corresponds to taking s very large, satisfying $s \gg |t|$ and large enough so that we can neglect the masses of the external particles. Without entering into details, the asymptotic behavior of the amplitudes in this limit can be captured to a large extent if one performs a partial wave expansion, and assumes that the partial wave coefficients can be analytically continued in the angular momentum l. If it is the case that there are only isolated poles in the l-plane, so-called Regge poles, whose positions $\alpha_i(t)$ may depend on t, then it can be shown that the form of the amplitude is determined in this limit by its behavior at these poles. More specifically,

$$\mathcal{A}(s, t) \sim \sum_i \frac{\eta_i + e^{-i\pi\alpha_i(t)}}{2} \beta_i(t) s^{\alpha_i(t)} , \tag{5.1.1}$$

where $\alpha_i(t)$ is called the Regge trajectory of the pole i, and η_i, which can be $+1$ or -1, is its signature[1].

When $s \to \infty$, the pole whose Regge trajectory $\alpha(t)$ has the largest real part will dominate and one finds

$$\mathcal{A}(s, t) \sim \frac{\eta + e^{-i\pi\alpha(t)}}{2} \beta(t) s^{\alpha(t)} , \tag{5.1.2}$$

providing a prediction for the high energy behavior of scattering amplitudes. In fact, it was precisely such behavior that motivated the early study of string theory as a theory for the strong interactions, since scattering amplitudes in string theory typically take this form, with

$$\alpha(t) = \alpha(0) + \alpha' t , \tag{5.1.3}$$

where α' is the called Regge slope and $\alpha(0)$ is the intercept.

[1] The reason for the introduction of the signature is technical and is related to the fact that the partial wave coefficients at even, and odd l have to be analytically continued separately in order for the continuation to be unique. If a given pole is found in the analytic continuation from even l then it has $\eta = 1$ and is said to have even signature, and vice versa. Since the signature factor can vanish at certain orders in perturbation theory, it may have important consequences.

Another application of Regge theory is to give the dependence of total cross sections on s, for large s. Invoking the optical theorem, the imaginary part of the forward scattering amplitude of a given process is related to the total cross section σ_{tot} by

$$2\mathrm{Im}\mathcal{A}_{aa}(s,\,0) = F\sigma_{tot}\ , \tag{5.1.4}$$

where $F = 2s$ if the masses of the incoming particles are negligible. Since there is no momentum transfer, the amplitude $\mathcal{A}_{aa}(s,\,0)$ should be evaluated in the Regge limit, if s is large. Equations (5.1.4) and (5.1.2) imply that

$$\sigma_{tot} \propto s^{\alpha(0)-1}\ , \tag{5.1.5}$$

where $\alpha(t)$ is the position of the leading Regge pole of the process. For examples of some early experimental checks of this behavior, see [212].

The interesting thing about (5.1.2) is that the amplitude can be interpreted as the exchange in the t-channel, as depicted in figure 5.1, of a single "particle", called a reggeon, of spin $\alpha(t)$. The reason for this interpretation is that in the Regge limit the components of the transferred momentum q^μ, with $q^2 = t$, are much smaller than the components of the momenta p_1 and p_2 of the incoming particles (which are of order $\sqrt{s}$). In fact, using that the incoming and outgoing particles are on-shell, it is easy to show that

$$q = \frac{t}{s}\,(p_2 - p_1) + q_\perp\ , \tag{5.1.6}$$

where $q_\perp$ is a transverse vector, i.e. orthogonal to p_1 and p_2, satisfying

$$q_\perp^2 \sim q^2 = t\ . \tag{5.1.7}$$

This implies that when calculating the coupling of a particle, exchanged in the t-channel, to the upper and lower particle lines, such as the coupling of the Reggeon in figure 5.1, we can use the eikonal approximation. This sets the momenta of the outgoing particles equal to the ingoing momenta on the corresponding lines, so the only available Lorentz vectors that can be used to couple to the exchanged particle are p_1 and p_2. If the exchanged particle has spin J, which we for simplicity take to be integer even though the half integer case is true as well, it has J Lorentz indices which must couple to the incoming particles. The upper line produces J factors of p_1 and the lower one J factors of p_2. Since the components of the incoming momenta are of order $\sqrt{s}$, the amplitude becomes proportional to s^J.

Of course, the reggeon is normally not a physical particle, since $\alpha(t)$ is in general complex and t-dependent. However, if there exists some physical particle of mass m and spin J with the same quantum numbers (apart from the spin) as a reggeon of trajectory $\alpha(t)$, then it is often the case that $\alpha(m^2) = J^2$. The reason is that if one considers

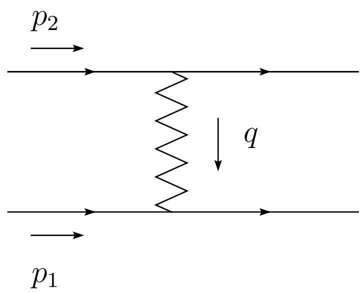

Figure 5.1: A reggeon exchange diagram.

the process obtained by crossing symmetry, where the reggeon is exchanged in the s-channel, one expects a resonance at $s = m^2$ corresponding to the physical spin J particle. Conversely, if for a given physical particle there exists a Regge trajectory for which $\alpha(m^2) = J^2$, one says that the particle reggeizes in the high-energy limit, and that the particle lies on the corresponding Regge trajectory.

It is not obvious, however, which particles should reggeize and which should not. In QED, for example, the electron reggeizes while the photon does not. It has been shown (see for example [214] for a review on these matters), however, that gauge particles reggeize if and only if the gauge group is semi-simple. In the $SU(2) \times U(1)$-sector of the standard model, for example, the W^+ and W^- reggeize, while the Z and photon do not. The reason for this is that both the Z and the photon contain the gauge boson B_μ of the original $U(1)$ group. For QCD and $\mathcal{N} = 4$, the gauge group is non-abelian, so we would expect the gluon to reggeize. In the next section we will see that this is indeed the case: the amplitude for color octet exchange in the t-channel behaves as $s^{\alpha(t)}$ in the Regge limit, where $\alpha(0) = 1$, showing that a massless, spin 1 particle with color octet quantum numbers, or in other words the gluon, lies on the corresponding Regge trajectory.

When QCD became established as the main theory for the strong interactions, it was then natural to ask if the Regge behavior of total cross sections and trajectories that had been observed experimentally could be derived through perturbation theory. Precisely such considerations led to the constructions that will be explained below, of the reggeized gluon, for color octet exchange, and the so-called BFKL Pomeron, for color singlet exchange. It may seem difficult to obtain (5.1.2) from a sum of Feynman diagrams, but the way that this behavior appears perturbatively is that, in the Regge limit, the scattering amplitudes receive, order by order in the perturbative expansion, logarithmic enhancements. Another well known context in which such logarithmic factors appear is that of Deep Inelastic Scattering, where collinear emission of gluons lead to logarithms of the form $(\alpha_S \log Q^2)^n$, where Q^2 is a transverse scale. These logarithms can be resummed, for instance by the introduction of the DGLAP equation[93], giving the evolution of the parton distribution functions with the scale.

In the Regge limit, the logarithms that appear do not depend on an external transverse scale Q^2, but rather the energy s, and it is upon resumming these logarithms that one finds a behavior of the form (5.1.2). This resummation is, in fact, necessary for large s since $\log s$ factors accompany the coupling α_S, making the order-by-order perturbation expansion ill-behaved. We will now turn to showing how this reggeization by resummation works out explicitly in the case of the gluon.

5.1.2 The Reggeized Gluon

As mentioned in the previous section, in the Regge limit of QCD we expect (given that the assumptions of Regge poles etc are satisfied) that the amplitude for color octet exchange can be described as the interchange of a Reggeon, the reggeized gluon, in the t-channel. From figure 5.1 and equation (5.1.2), we see that the reggeized gluon can be described as an ordinary gluon, but where the propagator (in Feynman gauge) has been changed from

$$D_{\mu\nu}(q) = -i\frac{g_{\mu\nu}}{q^2} \tag{5.1.8}$$

to

$$\tilde{D}_{\mu\nu}(q) = -i\frac{g_{\mu\nu}}{q^2}\left(\frac{s}{Q^2}\right)^{\alpha(t)-1} , \tag{5.1.9}$$

where $\alpha(t)$ is the Regge trajectory of the reggeized gluon, where Q^2 is a typical transverse momentum scale, and where the coupling of the gluon to the external particles include the signature factor, and any additional factors that may arise. The missing factor of s in (5.1.9) is provided, when forming the amplitude, by the momenta of the external particles.

We will now outline how the reggeized gluon is constructed in QCD perturbation theory. The way that this is done is to, order by order in the perturbative expansion, calculate the leading logarithmic approximation (LLA), which contains all terms with a maximum number of logarithms in s. Typically, at LLA one obtains an additional logarithm for each power of α_S. Upon resumming these logarithms one obtains the $s^{\alpha(t)}$-dependence. For a more detailed review of this construction, see [212].

For definiteness, let us consider a 2 quark to 2 quark amplitude in the Regge limit, with the exchange of color adjoint quantum numbers in the t-channel. The lowest order diagram for color octet exchange corresponds is shown in figure 5.2. This diagram does not contain any logarithmic enhancements but is still important since higher order diagrams will be proportional to it, and, just as was the case for the MHV amplitudes discussed in chapter 4, the helicity dependence is contained entirely in this tree level piece.

From equation (5.1.6) we see that all the components of the transferred momentum q are much smaller than the components of the incoming momenta p_1 and p_2, allowing

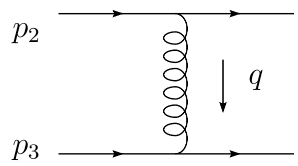

Figure 5.2: The tree level diagram for color octet exchange.

us to use the eikonal approximation, valid to first order in $\frac{t}{s}$, setting the momenta of the outgoing quarks equal to the momenta of the incoming quarks when evaluating the spinor products on the upper and lower fermion lines. The (outgoing) helicity of the outgoing quarks must also equal the helicity of the incoming quarks. Letting T^a_{ij} denote the quark color matrices, and h_2 the helicity, the upper line then contributes a factor

$$-ig_{YM}\bar{u}(h_2,\, p_2)\gamma^\mu u(h_2\, p_2)T^a_{ij} = -2ig_{YM}p_2^\mu T^a_{ij} \ . \tag{5.1.10}$$

We can here see the typical eikonal behavior of a vertex in which the only Lorentz index corresponds to the incoming momentum p_2. Inserting the gluon propagator $D_{\mu\nu}(q) = -i\frac{g_{\mu\nu}}{q^2} = -i\frac{g_{\mu\nu}}{t}$, the entire tree level amplitude becomes

$$\mathcal{A}_1 = (-i)4ig_{YM}^2 p_1 \cdot p_2 \frac{1}{t} H T^a \otimes T^a = 8\pi\alpha_S \frac{s}{t} H \tau^a \otimes \tau^a \ , \tag{5.1.11}$$

where we have used that $s = 2p_1 \cdot p_2$ in the Regge limit, where H encodes the helicity dependence (containing Kronecker-deltas setting ingoing and outgoing helicities to be the same),and where $T^a \otimes T^a$ is the tensor product of the representations of the color gauge group acting on the incoming quarks. Note the appearance of the $\frac{s}{t}$ factor.

But more than the tree level, we will be interested in the loop corrections since they will provide the logarithmic terms leading to reggeization. At one loop, the diagrams that will contribute in the Regge limit are shown in figure 5.3. Other contributions, such as vertex or propagator corrections will be subdominant. We will not explain why in detail, but a heuristic explanation is that such corrections are only sensitive to the momentum flowing in a reduced part of the diagram, while in order to receive enhancements in s one must add lines that connect both incoming particles. Another important omission is that of diagrams corresponding to an exchange of fermions or, in the case of $\mathcal{N} = 4$, scalar particles. This can be seen to be related to the discussion in the previous section, were it was explained that an exchange of particles of spin J will contribute factors of s^J in the Regge limit. Lower spin particles will therefore be suppressed with relation to the gluon, and only gluons will enter the Feynman diagrams. This explains why the (loop corrections) in the Regge limit of QCD must coincide with that of other gauge theories, including $\mathcal{N} = 4$ Super Yang Mills.

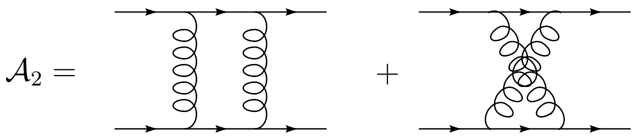

Figure 5.3: The one loop diagrams dominating in the Regge limit.

Instead of evaluating the diagrams of figure 5.3 directly, we can use unitarity based techniques, as was discussed in section 4.1, to find its imaginary part. There, we mentioned that the equation (4.1.17) could be implemented by using the Cutkosky rules, in which one writes down a Feynman diagram at the correct order, and "cuts" some intermediate propagators by putting them on-shell, thereby emulating the intermediate states of the right hand side of (4.1.17).

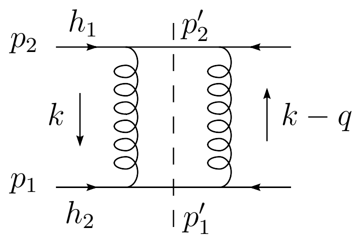

Figure 5.4: Evaluating the first one-loop diagram using the Cutkosky rules.

There is only one way to cut the first diagram of figure 5.3 so that the two subdiagrams on each side of the cut are connected, which is shown in figure 5.4, where p_1' and p_2' are the momenta of the cut propagators and $k = p_2 - p_2'$, while there is no way to cut the second diagram at this order implying that it does not contribute to the imaginary part of the amplitude. The fermion lines on the right hand side of the cut have been reversed due to the hermitian conjugation of the second amplitude in (4.1.17). In evaluating this integral we must integrate over all k such that p_1' and p_2' correspond to on-shell momenta. This is equivalent to introducing the delta-functions $\delta\left({p_1'}^2\right)\delta\left({p_2'}^2\right)$ in the one-loop integral, which thus reduces it from a four-dimensional to a two-dimensional integral. Also, because of these delta-functions, the integral over k will no longer be UV divergent.

In fact, the contributions to the amplitude for which k is of the order of $\sqrt{s}$ or larger will be negligible in the $s \to \infty$ limit, since the two gluonic propagators will then have very large denominators. In the Regge limit we can therefore take all of the components of k small, as compared to s, implying that both sides of the cut in figure 5.4 can themselves be evaluated as Regge limit amplitudes. We can then recycle our earlier result (5.1.11) for the tree level amplitude, and obtain

$$\mathrm{Im}\mathcal{A}_{2,s} = 32\pi^2\alpha_S^2 G_s H \int d(P.S.)\frac{s}{k^2}\frac{s}{(k-q)^2} \ , \tag{5.1.12}$$

where $G_s = (T^aT^b)\otimes(T^aT^b)$ is the color factor, and where $\int d(P.S)$ means an integral over the remaining two-dimensional phase-space. We have written the amplitude and the color factor with the subscript s, indicating that they correspond to the first diagram of figure 5.3, anticipating that we will obtain the second diagram by the crossing transformation $s \leftrightarrow u$.

To evaluate the integral in (5.1.12), and especially the integrals involved in higher order diagrams, it will be convenient to introduce Sudakov variables ρ, λ and $k_\perp$ through

$$k = \rho p_2 + \lambda p_1 + k_\perp \ , \tag{5.1.13}$$

where $k_\perp$ is transverse to p_1 and p_2. Instead of $k_\perp$ it is frequently more convenient to work with a 2-dimensional spatial vector $\boldsymbol{k}$, which is simply $k_\perp$ restricted to the transverse plane. From now on we will always take bold-faced letters to represent vectors in the transverse plane. Integrating over ρ and λ removes the on-shell delta-functions, and in the end one finds

$$\mathrm{Im}\mathcal{A}_{2,s} = 4\alpha_S^2 s G_s H \int d\boldsymbol{k}\frac{1}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2} = 8\pi\alpha_S\frac{s}{t}HG_s\frac{\alpha_S}{2\pi}\int d\boldsymbol{k}\frac{-\boldsymbol{q}^2}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2} \ . \tag{5.1.14}$$

Note that the propagators only depend on the transverse momenta, since in the Regge limit the contributions from the longitudinal components are negligible.

Having the imaginary part of the amplitude, one can deduce its real part by performing a contour integral in the s-plane. It is a standard result that the amplitude has branch cuts along the real axis corresponding to physical particle production (in either the s channel or crossed channels) for which the discontinuity of the amplitude along the cuts is given by the imaginary part it has slightly above the cut. Performing the integral around the branch cuts, one can apply Cauchy's integral formula to construct the full amplitude from its imaginary parts. In the Regge limit, the result simply corresponds to letting the full amplitude depend on

$$\log(-s) = \log s - i\pi \ . \tag{5.1.15}$$

Applied to (5.1.14), the relation implies that the leading logarithmic contribution to the first diagram of figure 5.3 is

$$\mathcal{A}_{2,s} = -8\pi\alpha_S\frac{s}{t}HG_s\log\left(\frac{-s}{Q^2}\right)\frac{\alpha_S}{2\pi^2}\int d^2\boldsymbol{k}\frac{-\boldsymbol{q}^2}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2} \ , \tag{5.1.16}$$

where Q^2 is a typical transverse scale for the process under study, inserted for dimensional reasons. The choice of Q^2 is irrelevant for the leading logarithm contribution, since a different scale Q'^2 would produce

$$\log\left(\frac{s}{Q'^2}\right) = \log\left(\frac{s}{Q^2}\right) + \log\left(\frac{Q^2}{Q'^2}\right) \ , \tag{5.1.17}$$

and the second term can be discarded at LLA. The scale dependence is, however, highly relevant for higher order corrections (see for example [213]).

The second diagram in figure 5.3 is now obtained from the first one by performing a crossing transformation interchanging u and s, and changing the color factor from G_s to $G_u = (T^aT^b) \otimes (T^bT^a)$, giving

$$\mathcal{A}_{2,u} = -8\pi\alpha_S \frac{u}{t} H G_u \log\left(\frac{-u}{Q^2}\right) \frac{\alpha_S}{2\pi^2} \int d^2\boldsymbol{k} \frac{-\boldsymbol{q}^2}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2} \ . \tag{5.1.18}$$

Since $s+t+u$ is the sum of the masses of the ingoing and the outgoing particles, and these are negligible in the Regge limit, we have $u \approx -s$, implying that (5.1.18) is purely real, as we saw earlier from the impossibility of cutting the Feynman diagram into two connected pieces.

Now, since the real parts of the amplitudes (5.1.16) and (5.1.18) have one more logarithm than the imaginary part (5.1.14), as long as the real part does not vanish, at leading logarithmic order the imaginary part can be discarded, and we are left with

$$\mathcal{A}_2 = -8\pi\alpha_S \frac{s}{t} H (G_s - G_u) \log\left(\frac{s}{Q^2}\right) \frac{\alpha_S}{2\pi^2} \int d^2\boldsymbol{k} \frac{-\boldsymbol{q}^2}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2} \ . \tag{5.1.19}$$

Straightforward manipulations give the total color factor as

$$G_s - G_u = (T^aT^b) \otimes ([T^a, T^b]) = \cdots = -\frac{N}{2} T^a \otimes T^a \ , \tag{5.1.20}$$

Note that we are indeed only left with a color structure corresponding to an exchange of gluon quantum numbers. If we project onto color singlet exchange, as we will do in the next section, the real part does vanish, however, and the imaginary part becomes important.

Putting everything together, and using the expression (5.1.11) for the tree level amplitude, we have

$$\mathcal{A}_2 = \mathcal{A}_1 \omega(t) \log\left(\frac{s}{Q^2}\right) \ , \tag{5.1.21}$$

where

$$\omega(t) = -\frac{g^2}{\pi} \int d^2\boldsymbol{k} \frac{t}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2} \ , \tag{5.1.22}$$

which only depends on $t = -\boldsymbol{q}^2$, due to the symmetry of (5.1.22) under rotations of $\boldsymbol{q}$, and where the planar coupling

$$g^2 = \frac{g_{YM}^2 N}{16\pi^2} = \frac{\alpha_S N}{4\pi} \tag{5.1.23}$$

is defined as before. Setting $\alpha(t) = 1 + \omega(t)$, we can interpret this as the Regge trajectory of the reggeized gluon, since, including higher order diagrams, which we will briefly outline next, the complete amplitude becomes

$$\mathcal{A} = \mathcal{A}_1 \left(\frac{s}{Q^2}\right)^{\alpha(t)-1} \frac{1 - e^{i\pi\alpha(t)}}{2} , \tag{5.1.24}$$

and so correspond to a Regge trajectory of odd signature. The signature factor is actually subleading in the Regge limit, since the powers of the coupling in its expansion are not accompanied by $\log s$ enhancements. At the first two orders, the signature factor is

$$1 - i\frac{\pi}{2}\omega(t) , \tag{5.1.25}$$

reproducing the discarded imaginary part (5.1.14) (if the color factor G_s is projected onto the octet, giving the additional factor of $\frac{1}{2}$).

There are two things to note about (5.1.22). Firstly, the integral over $\boldsymbol{k}$ is infra-red divergent. This is not a problem in real life, however, since then the reggeized gluon does not couple to on-shell quarks. Instead one must introduce so-called "impact factors" which describe the coupling of the reggeon to a given object, for example a hadron. Infra-red safe expressions are then obtained if the impact factors suppress the coupling of the reggeon for low k. Secondly, it is not difficult to see that if one regularizes the integral by dimensional regularization, performing it in $2+\epsilon$ dimensions, it will be proportional to $(-t)^{\varepsilon/2}$, implying that

$$\alpha(0) = 1 . \tag{5.1.26}$$

This confirms the identification of the discovered Regge trajectory as that of the reggeized gluon since a massless, spin 1 particle lies on the trajectory.

We will now briefly outline which type of diagrams that appear at higher orders in the perturbative expansion. The archetypical type of diagram contributing in the Regge limit is the ladder diagram of which the (cut) two-loop instance is shown in figure 5.5. As before, both sides of the diagram can be evaluated in the Regge limit, and the fermion-gluon vertices can be calculated in the eikonal approximation, so the upper line will produce a factor of p_1^μ, while the lower line gives p_2^ν. The evaluation is therefore straightforward. Again, it proves convenient to introduce Sudakov variables for the momenta k_1 and k_2:

$$k_i = \rho_i p_2 + \lambda_i p_1 + k_{i\perp} , \tag{5.1.27}$$

and it turns out that the leading logarithmic contribution to the diagram 5.5 comes from the region of phase space in which

$$\begin{aligned} \rho_2 \ll \rho_1 \ll 1 \\ |\lambda_1| \ll |\lambda_2| \ll 1 \ . \end{aligned} \tag{5.1.28}$$

The logarithms of the energy thus appear when one has a strict ordering of the longitudinal momenta.

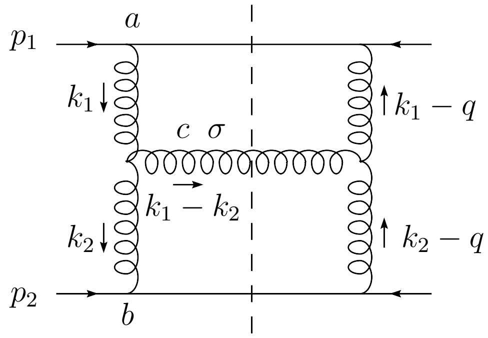

Figure 5.5: The two-loop ladder diagram.

In some theories, such as scalar theories, only ladder diagrams will contribute in the Regge limit. This is not the case for gauge theories, though. For example, if the left side of the horizontal gluon were attached to one of the quarks instead of the vertical gluon, it is true that an additional hard propagator would appear along the fermion line. However, we have an extra factor of momenta for each gluon three vertex or eikonal gluon-fermion vertex, implying that such diagrams can be important as well. Together with the left side of the cut of the ladder diagram we must in fact sum over the subdiagrams shown in figure 5.6.

The phase space integral for those alternate subdiagrams is the same as for the ordinary diagram, and the leading logarithmic approximation will still correspond to the ordering (5.1.28) in longitudinal momenta. There will, of course, also be corresponding alternatives to the right side of the ladder diagram, which can be combined with the left side in all possible ways, implying a total of 25 diagrams. Fortunately, these diagrams can be presented in a way that simplifies things somewhat, which becomes necessary at higher orders. In the Regge limit, the subdiagrams (a) and (b) turn out to be of equal magnitude, but opposite sign. This implies that the color factor for (a) and (b) becomes

$$(T^bT^c - T^cT^b) \otimes T^b = [T^b, T^c] \otimes T^b = f_{abc}T^a \otimes T^b \ , \tag{5.1.29}$$

which is the same as the color factor from the ordinary ladder diagram of figure 5.5. The same is true for the subdiagrams (c) and (d). This allows us to define a "non-local" vertex

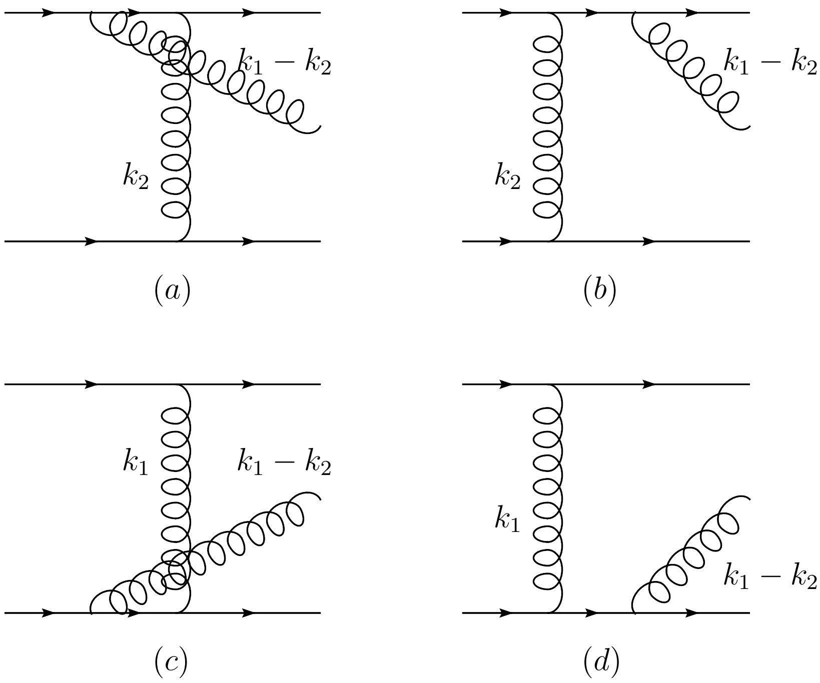

Figure 5.6: Alternate subdiagrams for the left hand side of the ladder diagram.

$\Gamma^{\sigma}_{\mu\nu}$, known as the Lipatov vertex, which makes the diagram of figure 5.7 contribute the same as the sum of the diagrams of figure 5.6 and the left side of 5.5.

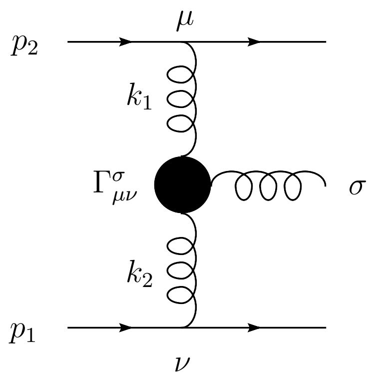

Figure 5.7: The non-local vertex Γ, adding up the contributions of several subdiagrams.

The reason that Γ is called a non-local vertex is that it will, just like the propagator of a reggeon, not only depend on the momenta flowing through it, but also other momenta, in this case p_1 and p_2. Calculating the contributing subdiagrams, one obtains

$$\Gamma^{\sigma}_{\mu\nu} = g_{YM} \frac{2p_{2\mu}p_{2\nu}}{s} \left[\left(\rho_1 + \frac{2\boldsymbol{k}_1^2}{\lambda_2 s} \right) p_2^{\sigma} + \left(\lambda_2 + \frac{2\boldsymbol{k}_2^2}{\rho_1 s} \right) p_1^{\sigma} - (k_1 + k_2)_{\perp}^{\sigma} \right] . \qquad (5.1.30)$$

Here, the terms not containing factors of $\boldsymbol{k}_1^2$ or $\boldsymbol{k}_2^2$ come from the left part of the ladder diagram, while the terms with the squares of transverse momenta come from the diagrams of figure 5.6. The $\boldsymbol{k}_1^2$ or $\boldsymbol{k}_2^2$ appear because it is necessary to multiply and divide the subdiagrams of figure 5.6 by those terms in order to make up for the missing propagator denominators that are present in figure 5.7. This implies that the two-loop diagrams discussed so far are all included in the effective ladder diagram shown in figure 5.8.

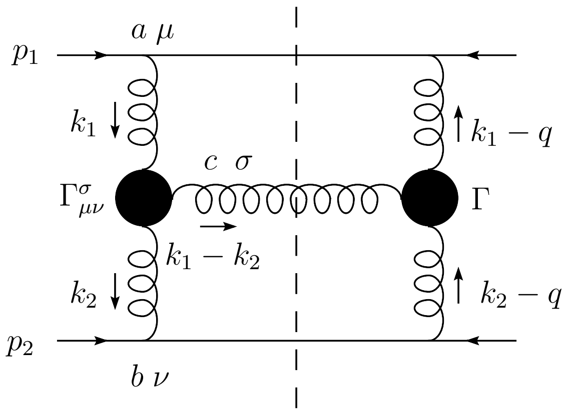

Figure 5.8: The effective two-loop ladder diagram.

Now, apart from the effective ladder diagram, the remaining LLA two-loop contributions to color octet exchange are of the type shown in figure 5.9. They simply consist of putting the tree amplitude (5.1.11) on one side of the cut, and the one-loop amplitude (5.1.21) on the other side. Since the lower order amplitudes have already been calculated, one can calculate this second class of diagrams using the Cutkosky rules.

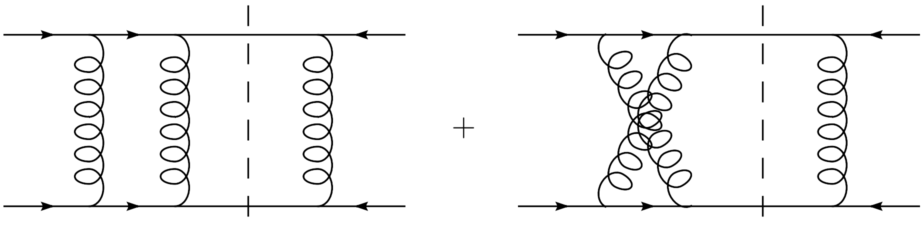

Figure 5.9: Two-loop diagrams contributing in the Regge limit that are not included in the effecive ladder diagram. Not shown are the analogous diagrams with one gluon to the left of the cut.

It is now straightforward, although a bit tedious, to evaluate the diagrams in figures 5.8 and 5.9, and then adding the crossed contribution obtained from taking $s \leftrightarrow u$, to obtain the imaginary part of the two-loop octet exchange amplitude $\mathcal{A}_3$. The result is

$$\mathrm{Im}\mathcal{A}_3 = -\frac{N^2\alpha_S^2}{16\pi^3}\mathcal{A}_1 \int d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2 \int_{\boldsymbol{k}_2^2/s}^{1} \frac{d\rho_1}{\rho_1} \frac{\boldsymbol{q}^2\boldsymbol{q}^2}{\boldsymbol{k}_1^2\boldsymbol{k}_2^2(\boldsymbol{k}_1-\boldsymbol{q})^2(\boldsymbol{k}_2-\boldsymbol{q})^2} \ . \tag{5.1.31}$$

As argued above, the ρ_1 integral appears as part of the phase space integration when the momenta are longitudinally ordered, as in (5.1.28). It gives a $\log\left(\frac{s}{k_2^2}\right)$ factor, which can be taken as $\log\left(\frac{s}{Q^2}\right)$, at LLA order, as was argued earlier when calculating the one-loop amplitude. Using (5.1.22) and noting that the integrals over the transverse momenta factorize, we have

$$\mathrm{Im}\mathcal{A}_3 = -\mathcal{A}_1 \omega(t)^2 \log\left(\frac{s}{Q^2}\right) . \tag{5.1.32}$$

And using (5.1.15), the real part of the amplitude, which we can take as the entire amplitude at LLA, is

$$\mathrm{Re}\mathcal{A}_3 = \frac{\mathcal{A}_1}{2} \omega(t)^2 \log^2\left(\frac{s}{Q^2}\right) . \tag{5.1.33}$$

This is the third order term in the expansion of (5.1.24). We thus see that the reggeization trend continues beyond two loops.

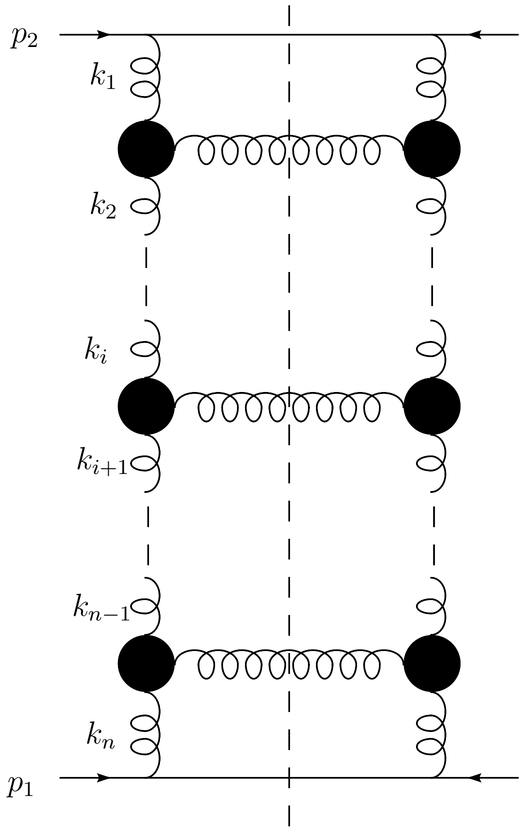

Figure 5.10: Higher order effective ladder diagrams contributing at LLA.

As we go to higher orders, the typical contribution to the LLA amplitude is again effective ladder diagrams, shown in figure 5.10. A phase space analysis then shows that

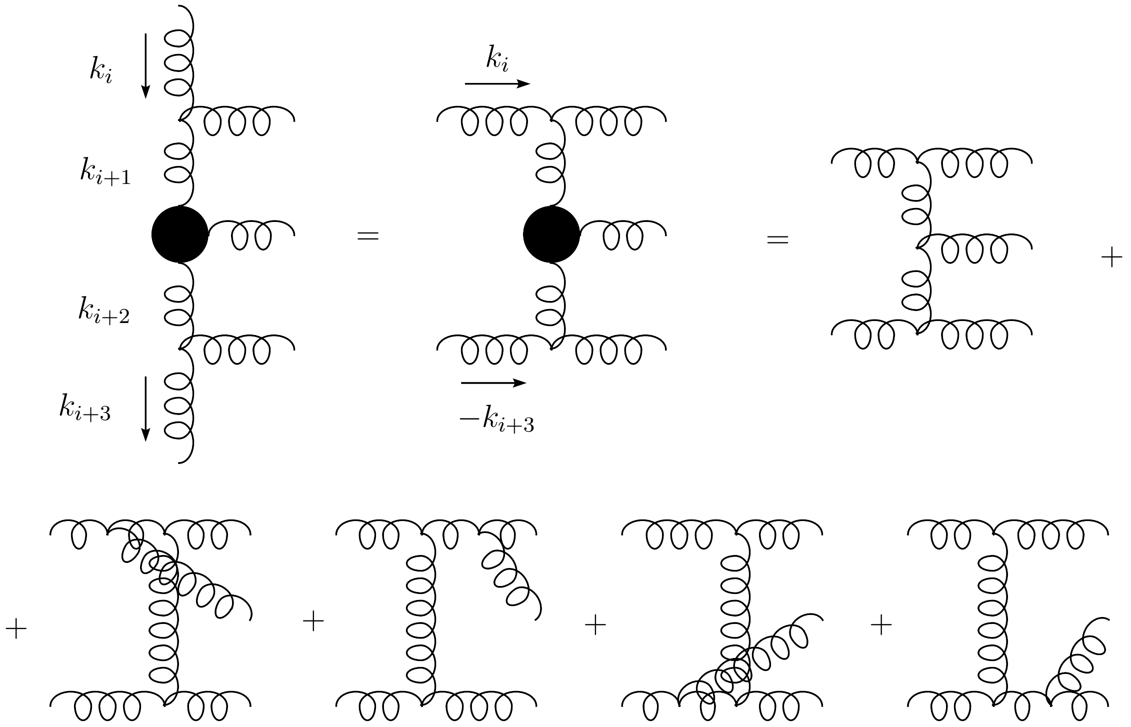

Figure 5.11: The definition of the non-local vertex for gluons in a ladder diagram.

once again the relevant kinematical region corresponds to the longitudinal ordering

$$\begin{aligned} &\rho_n \ll \rho_{n-1} \ll \cdots \ll \rho_2 \ll \rho_1 \ll 1 \\ &|\lambda_1| \ll |\lambda_2| \ll \cdots \ll |\lambda_{n-1}| \ll |\lambda_n| \ll 1 \ . \end{aligned} \tag{5.1.34}$$

The definition of the non-local vertex $\Gamma^{\sigma}_{\mu\mu}$ between gluons is similar to the definition given in the last section for fermions, and is shown pictorially in figure 5.11. And in fact, due to the longitudinal ordering of the momenta, the eikonal approximation is valid even in this case, which implies that the Lipatov vertex is given by equation (5.1.30) here as well[2]. The definition of two adjacent Lipatov vertices may seem unclear, but it can be shown rigorously that the tree-level amplitude for 2 particles to 2 particles + $n-1$ gluons, in the so-called Multi-Regge limit, corresponding to the kinematical regime (5.1.34), is indeed given by evaluating the left side of figure 5.10 using (5.1.30) for the vertices [215].

The main complication stems from having loop diagrams on one or both sides of the cut, such as the two-loop diagrams in figure 5.9, as such diagrams are not included in the effective ladders. But due to the large number of possible diagrams at high loop order it would thus seem difficult to calculate the all-order LLA amplitude. Fortunately, it is

[2]It would seem that we should obtain a different answer since we are replacing the fermion propagator of the diagrams in figure 5.6 with a gluon propagator. When the eikonal approximation is valid, however, the two diagrams give the same result.

possible to construct the reggeized gluon recursively, in such a way that its perturbative expression at a given order is determined from its expansion at lower orders. The construction is shown in figure 5.12, where a reggeized gluon is drawn as an ordinary gluon with a dash through it. The non-local vertices in the figure should be evaluated using the same expression (5.1.30) as before, and the signature factors should be omitted in the coupling of the reggeized gluons to the quarks in the right hand side. The reggeized gluons on the right hand side thus behave like ordinary gluons, but with the reggeized gluon propagator [3]. The prescription is thus that the imaginary part of the reggeized gluon is given by a sum of effective ladder diagrams where the vertical propagators themselves are given by reggeized gluons, together with the crossed $s \leftrightarrow u$ contribution.

At this point the recursive construction is only an Ansatz, but assuming that the gluon reggeizes up to a certain order in such a way that its effective propagator is given by (5.1.9), which we have seen that it does up to two loops, then one can evaluate the right hand side of the equation shown in the figure, and obtain an expression for the propagator valid one order higher in perturbation theory. If that expression coincides with what one obtains from the left hand side, again using the assumed form of the reggeized gluon propagator, then it follows that the iterative construction indeed produces an expansion that is reggeized to any perturbative order, with a propagator that coincides with (5.1.9). It does not follow, however, that one thereby captures the entire LLA, but if the complete amplitude reggeizes, as one would expect from Regge theory, and the first perturbative orders, this would indeed be the case.

We will now outline how one shows that the right and left hand sides of figure 5.12 coincide. To begin with, in the kinematic regime (5.1.34), corresponding to the leading logarithmic approximation, the reggeized gluon propagators take a simple form in terms of the Sudakov variables. From (5.1.9) we have that the i:th propagator of figure 5.12 is given by

$$\tilde{D}_{\mu\nu} = i\frac{g_{\mu\nu}}{\boldsymbol{k}_i^2}\left(\frac{\hat{s}_i}{Q^2}\right)^{\omega(k_i^2)} , \tag{5.1.35}$$

where $\hat{s}_i = (k_{i-1} - k_{i+1})^2$. In Sudakov variables this is

$$\hat{s}_i = (k_{i-1} - k_{i+1})^2 = -s\rho_{i-1}\lambda_{i+1} - (\boldsymbol{k}_{i-1} - \boldsymbol{k}_{i+1})^2 . \tag{5.1.36}$$

The last term can be dropped at LLA, since it corresponds to swapping logarithms in s for logarithms in the transverse momenta, while the first term can be rewritten as

$$\frac{\rho_{i-1}}{\rho_i}(-s\rho_i\lambda_{i+1}) = \frac{\rho_{i-1}}{\rho_i}(\boldsymbol{k}_i - \boldsymbol{k}_{i+1})^2 , \tag{5.1.37}$$

[3]The author must confess his ignorance as to how the right hand side could be expanded perturbatively, beyond the first few orders. The justification of the right hand side can be made by other means, than the purely diagrammatic ones, though, using special techniques from Regge theory [216].

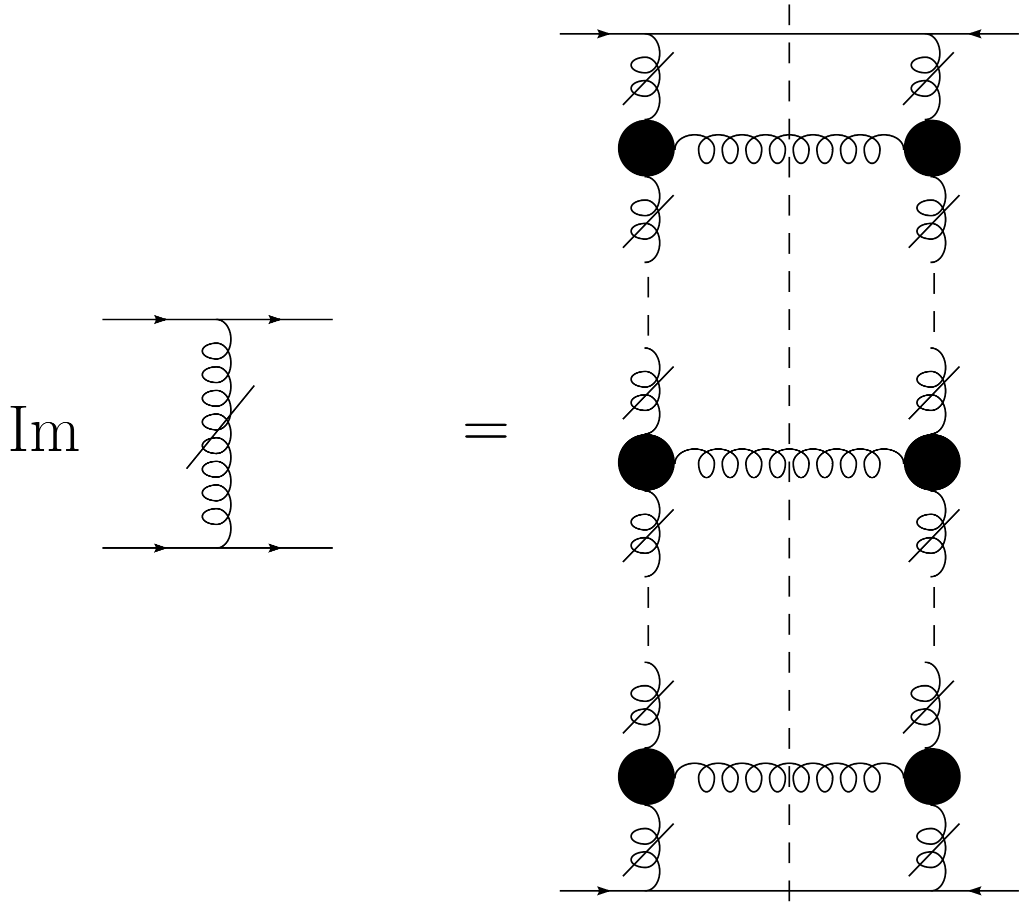

Figure 5.12: The iterative construction of the (imaginary part of the) reggeized gluon, where one sums over the number of rungs of the ladder, and then adds the crossed contribution. The reggeized gluons on the right hand side behave as ordinary gluons (in the way that they couple to other particles), except for their modified propagators.

where we for the last equality have used the one shell condition, $(k_i - k_{i-1})^2 = 0$ for the i:th horizontal gluon, written in Sudakov variables. So,

$$\frac{\hat{s}_i}{(\boldsymbol{k}_i - \boldsymbol{k}_{i+1})^2} = \frac{\rho_{i-1}}{\rho_i} \ . \tag{5.1.38}$$

Once again, at LLA the scale is irrelevant, and we can substitute $(\boldsymbol{k}_i - \boldsymbol{k}_{i+1})^2$ in the previous equation for Q^2. We thus have

$$\tilde{D}_{\mu\nu} = i\frac{g_{\mu\nu}}{\boldsymbol{k}_i^2}\left(\frac{\rho_{i-1}}{\rho_i}\right)^{\omega(-\boldsymbol{k}_i^2)} \quad . \tag{5.1.39}$$

Let us now move on to the phase space integral for a ladder with vertical momenta

$k_1 \ldots, k_n$, which, when written directly in Sudakov variables is

$$\int (P.S.) = \frac{s^n}{2^{4n-1}\pi^{3n-1}} \int \prod_{i=1}^{n} d\rho_1 d\lambda_i d^2\boldsymbol{k}_i \prod_{j=1}^{n-1} \delta(s(\rho_j - \rho_{j+1})(\lambda_j - \lambda_{j+1}) - (\boldsymbol{k}_j - \boldsymbol{k}_{j+1})^2) \times$$

$$\delta(-s(1-\rho_1)\lambda_1 - \boldsymbol{k}_1^2)\delta(s(1+\lambda_n)\rho_n - \boldsymbol{k}_n^2) \ . \tag{5.1.40}$$

Imposing the longitudinal ordering (5.1.34), and performing the λ_i integrals (which will produce a $1/s\rho_{i-1}$ factor for each λ_i), this simplifies to[4]

$$\int (P.S.) = \frac{1}{2^{4n-1}\pi^{3n-1}} \prod_{i=1}^{n-1} \int_{\rho_{i+1}}^{1} \frac{d\rho_i}{\rho_i} \int_0^{\rho_{n-1}} d\rho_n \int \prod_{j=1}^{n} d^2\boldsymbol{k}_j \, \delta(s\rho_n - \boldsymbol{k}_n^2) \ . \tag{5.1.41}$$

Considering that the ρ_i integrals are nested, and that the integrand contains the factors (5.1.39), it would seem that they would be difficult to perform. Fortunately, there is a transform, the Mellin transform $\mathbb{M}$, already introduced in equation (2.5.20) in the context of analytical continuations of anomalous dimensions, which unravels the integrals. The Mellin transform $\mathcal{F}(\omega)$ of a function $f\left(\frac{s}{Q^2}\right)$ is defined as

$$\mathcal{F}(\omega) = \int_1^\infty d\left(\frac{s}{Q^2}\right)\left(\frac{s}{Q^2}\right)^{-\omega-1} f\left(\frac{s}{Q^2}\right) \ , \tag{5.1.42}$$

where ω is not the same as $\omega(t)$, but where we choose to keep this rather confusing notation for historical reasons. The logic behind the notation will be clear shortly as we will see that $\omega(t)$ will define a pole in the ω-plane. The transforms that are relevant to us are

$$\mathbb{M}\left[\log^r\left(\frac{s}{Q^2}\right)\right] = \frac{r!}{\omega^{r+1}} \ , \tag{5.1.43}$$

$$\mathbb{M}\left[\left(\frac{s}{Q^2}\right)^\alpha\right] = \frac{1}{\omega-\alpha} \ , \tag{5.1.44}$$

and notably, if

$$f(s) = Q^2 \prod_{i=1}^{n} \int_{\rho_{i+1}}^{1} \frac{d\rho_i}{\rho_i} f_i\left(\frac{\rho_{i-1}}{\rho_i}\right) \delta(s\rho_n - Q^2) \ , \tag{5.1.45}$$

its Mellins transform is given by

$$\mathcal{F}(\omega) = \prod_{i=1}^{n} \mathcal{F}_i(\omega) \ , \tag{5.1.46}$$

where $\mathcal{F}_i(\omega)$ is the Mellin transform of $f_i\left(\frac{s}{Q^2}\right)$.

[4] The upper integration limits of the ρ_i integrals have been taken to be 1, which is equivalent to ρ_{i-1}, since this gives the same integration region.

The equation (5.1.44) provides an interpretation for ω, since we see that poles in the ω plane will correspond to Regge trajectories of the amplitude. And since the Regge trajectories appear as poles in the analytical continuation of the amplitude in the partial-wave expansion, ω can be considered a complex angular momentum.

Let us now define a Mellin space amplitude $\mathcal{F}(\omega,\, \boldsymbol{k},\, \boldsymbol{q})$ as the Mellin transform of the imaginary part of the ladder of reggeized gluons $\mathcal{A}_{Rg}(s,\, t)$, omitting the tree-level factor, and extracting an integral over the transverse momenta of the lowest rung in the ladder. More precisely

$$\int d\left(\frac{s}{Q^2}\right)\left(\frac{s}{Q^2}\right)^{-\omega-1}\left(\frac{\mathrm{Im}\mathcal{A}_{Rg}}{\mathcal{A}_1}\right) = \int \frac{d^2\boldsymbol{k}}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2}\mathcal{F}(\omega,\, \boldsymbol{k},\, \boldsymbol{q})\ . \tag{5.1.47}$$

The reason that we choose to extract the transverse momentum integral is that, as we will see shortly, it simplifies the calculation of $\mathcal{F}$ considerably.

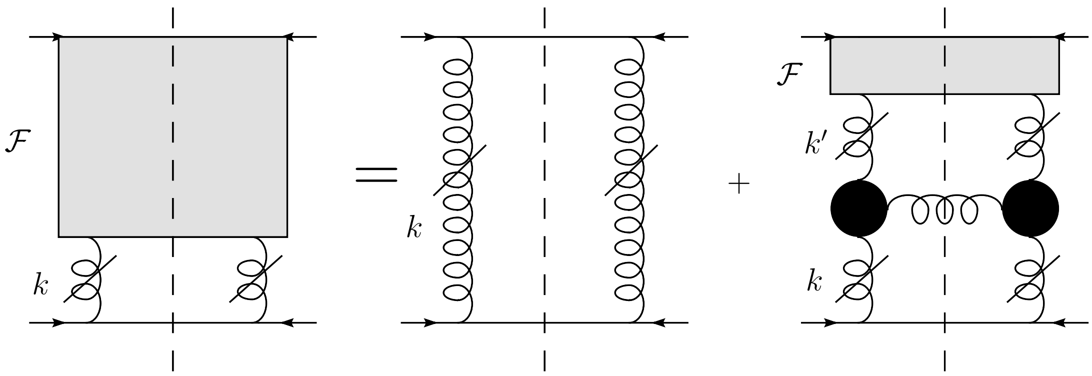

Figure 5.13: The integral equation that resums the reggeized gluon ladder diagrams that recursively generate the reggeized gluon itself. The momenta k' is integrated over.

We can then resum the entire ladder by solving the integral equation shown pictorially in figure 5.13, and which written out is

$$\begin{aligned}\mathcal{F}(\omega,\, \boldsymbol{k},\, \boldsymbol{q}) = &\frac{\pi}{2}\frac{4g^2}{\pi}\frac{\boldsymbol{q}^2}{\omega-(\omega(-\boldsymbol{k}^2))-(\omega(-(\boldsymbol{k}-\boldsymbol{q})^2))}\\ &-\frac{4g^2}{\pi}\int d^2\boldsymbol{k}'\frac{\mathcal{F}(\omega,\, \boldsymbol{k}',\, \boldsymbol{q})}{\omega-\omega(-\boldsymbol{k}^2)-\omega(-(\boldsymbol{k}-\boldsymbol{q})^2)}\\ &\frac{1}{\boldsymbol{k}'^2(\boldsymbol{k}'^2-\boldsymbol{q})^2}\left(\boldsymbol{q}^2-\frac{\boldsymbol{k}^2(\boldsymbol{k}'-\boldsymbol{q})^2+\boldsymbol{k}'^2(\boldsymbol{k}-\boldsymbol{q})^2}{(\boldsymbol{k}-\boldsymbol{k}')^2}\right)\ .\end{aligned} \tag{5.1.48}$$

All of the terms are easily identifiable, if one has (5.1.44) and the propagator of the reggeized gluon in mind, except perhaps the last parenthesis, which is the result of contracting the two Lipatov vertices. The merit of going to Mellin space is clear as the integrals over the ρ_i have unraveled, the contributions from the different rungs in the ladder factorize, and we are only left with the integrals over the transverse momenta. It should be noted, though, that the color factors obtained from the diagrams in the integral equation are actually not proportional to the color factor $T^a \otimes T^a$ of $\mathcal{A}_1$, but only becomes so after adding the u-channel contribution. For simplicity, however, in writing down the integral equation, the color factor is taken to be half that which is obtained after addition of the crossed contribution. The final result is, of course, the same. Let us also note that if we had included the transverse momentum integrals of (5.1.47) in the definition of $\mathcal{F}$, the integral equation would have been much messier since we would, for example, have to integrate over the arguments of the Regge trajectories.

Anticipating the answer, the integral equation is solved by the making the ansatz that its solution should not depend on k. The ansatz turns out to be correct, with the result that

$$\mathcal{F}(\omega,\, \boldsymbol{k},\, \boldsymbol{q}) = \frac{\pi}{2}\frac{4g^2\boldsymbol{q}^2}{\pi}\frac{1}{\omega - \omega(-\boldsymbol{q}^2)}\ . \tag{5.1.49}$$

Inserting this into (5.1.47), and using the inverse of (5.1.44), together with the trajectory (5.1.22) for the reggeized gluon,

$$\mathrm{Im}\mathcal{A}_{Rg}(s,\, t) = -\frac{\pi}{2}\omega(t)\left(\frac{s}{Q^2}\right)^{\alpha(t)-1}\mathcal{A}_1\ , \tag{5.1.50}$$

which, using (5.1.15), is seen to be the imaginary part of

$$\mathcal{A}_{Rg}(s,\, t) = \frac{1}{2}\left(\frac{-s}{Q^2}\right)^{\omega(t)}\mathcal{A}_1\ . \tag{5.1.51}$$

Adding the u-channel contribution we finally obtain that the amplitude for quark-quark scattering, interchanging a reggeized gluon, is

$$\mathcal{A}(s\, t) = \mathcal{A}_1\left(\frac{s}{Q^2}\right)^{\alpha(t)-1}\frac{1 - e^{i\pi\alpha(t)}}{2}\ . \tag{5.1.52}$$

Since this coincides with the proposed reggeized gluon expression, this justifies the recursive construction. The obtained amplitude also includes the signature factor. This might seem slightly unnecessary since it is not a part of the LLA. It does, however, give the LLA of the imaginary part of the amplitude, and as we will see is the case for the pomeron in the next section, it can be most relevant if it were to vanish at some perturbative orders.

5.1.3 BFKL and the Pomeron

After the long discussion on the reggeized gluon, we will now turn to the calculation of the exchange amplitude for the reggeon responsible for color singlet exchange, the pomeron. Actually, we require something more of the Pomeron than color singlet exchange. We require that it does not transmit any charge whatsoever, and thus has the quantum numbers of the vacuum. Perturbatively, this can be achieved through the same diagrammatic expansion as the one giving the reggeized gluon, but where one projects out the color singlet contribution to each diagram. Since the integrations over phase space will be the same in both cases, the LLA will correspond to the same ordering of longitudinal momenta. For these reasons, the discussion of color singlet exchange will be brief, and we will arrive at the corresponding integral equation, the famous BFKL equation[10], almost directly.

There will be some significant differences with the reggeized gluon, however.

1. The different color factors involved imply that the different diagrams contributing to the two-loop amplitude, given earlier in figures 5.8 and 5.9, will no longer add up to give a single Regge pole, with a well defined Regge trajectory. Instead, as we will see shortly, the solution to the BFKL equation at LLA will be a branch cut. In QCD, when higher order corrections are introduced, the branch cut discretizes, and the amplitude is again described by single Regge poles. There is not a single pole, however, and the pomeron is instead defined to correspond to the leading Regge pole, that is, the pole that dominates at large s. At LLA, one instead defines the pomeron as a reggeon with a regge trajectory given by the branch point of the branch cut, and will again dominate at large center-of-mass energies.

2. There is no tree level amplitude for pomeron exchange since there is no elementary particle in QCD having the quantum numbers of the vacuum. Instead, the first contributing diagrams will be the one-loop graphs in figure 5.3. The way that this behavior is incorporated in the form (5.1.2) for reggeon exchange is to assume that the pomeron has even signature. The signature factor is then

$$\frac{1}{2}\left(1+e^{i\pi\alpha_P(t)}\right) \ , \tag{5.1.53}$$

   which, if the Regge trajectory takes the form $\alpha_P(t) = 1 + \mathcal{O}(\alpha_S)$, is zero at first order, and purely imaginary at one-loop. The reason that the one-loop amplitude becomes purely imaginary was touched on in the previous section: when adding the contributions from the s and u-channels to the real part of the amplitude, a color factor corresponding to pure octet exchange is obtained.

3. Even though, as we will see, the BFKL equation is constructed using reggeized gluons, the IR divergences of the gluon Regge trajectory will cancel out. The BFKL equation is therefore well behaved in the IR limit.

Let us now turn to the construction of the BFKL equation. We are interested in resumming ladder diagrams of the form of the right hand side of 5.12, but where we project out the singlet contributions at the upper and lower ends of the ladder. This implies that the only difference between this case and that of the reggeized gluon lies in the color factors. For the term where two reggeized gluons are interchanged, the original color factor, in the s-channel, was $(T^aT^b) \otimes (T^aT^b)$. The way to project out the singlet at one of the ends is to simply take the trace $\frac{1}{N}\mathrm{Tr}(\cdot)$. This gives the color factor for the exchange of two reggeized gluons projected onto the singlet as

$$C_0 \equiv \frac{1}{N^2}\mathrm{Tr}(T^aT^b)\mathrm{Tr}(T^aT^b) = \frac{1}{4N^2}\delta_{ab}\delta_{ab} = \frac{N^2-1}{4N^2} \ , \tag{5.1.54}$$

where we have used that the T^a are normalized such that $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta_{ab}$ and that the number of generators in the adjoint representation of $SU(N)$ is N^2-1. When we add an additional rung to the ladder, the color factor gets multiplied by N, since the non-local vertices (5.1.30) contribute one instance of the structure constants f_{abc}, and

$$\delta_{cd}f_{cae}f_{dbe} = N\delta_{ab} \ . \tag{5.1.55}$$

This should be contrasted with the case of color singlet exchange, where each additional rung provided a factor of $N/2$.

In order to unravel the nested integrals of the ρ_i, we can again perform the Mellin transform of the singlet exchange amplitude $\mathcal{A}^0$. This time, as we will explain shortly, we want to factor out not only an integral over the transverse momentum $\boldsymbol{k}_1$ at one end of the ladder, but also an integral over a transverse momentum k_2 at the other end. We thus define $F(\omega,\, \boldsymbol{k}_1,\, \boldsymbol{k}_2,\, \boldsymbol{q})$ by

$$\int d\left(\frac{s}{Q^2}\right)\left(\frac{s}{Q^2}\right)^{-\omega-1}\mathcal{A}^0 = 4is\alpha_S^2 HC_0 \int \frac{d^2\boldsymbol{k}_1 d^2\boldsymbol{k}_2}{\boldsymbol{k}_2^2(\boldsymbol{k}_1-\boldsymbol{q})^2}F(\omega,\, \boldsymbol{k}_1,\, \boldsymbol{k}_2,\, \boldsymbol{q}) \ , \tag{5.1.56}$$

where we have extracted the factor $4is\alpha_S^2HC_0$ (just as we factored out $\mathcal{A}_1$ in the case of the reggeized gluon), and the propagator denominators for convenience.

One of the reasons for factoring out the momentum integrals at both sides of the ladder is that we then can interpret F as a 4-point reggeized gluon Greens function.From now on we will often refer to F as simply the Greens function. In fact, this interpretation is used when deriving the relationship between BFKL and the anomalous dimensions of the $SL(2)$ sector of $\mathcal{N}=4$ Super Yang Mills, discussed in section 2.5.1

Comparing with (5.1.48), accounting for the different color factors (letting $N \to 2N$) and the different numerical factors and propagator denominators included in the definition of F, we can directly write down an integral equation, shown in figure 5.14, for color singlet

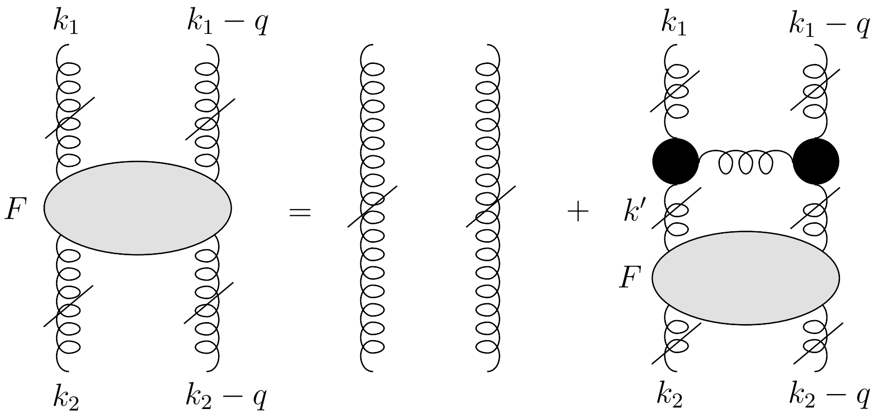

Figure 5.14: The integral equation determining the LLA contribution to color singlet exchange.

exchange:

$$F(\omega,\,\boldsymbol{k}_1,\,\boldsymbol{k}_2,\,\boldsymbol{q})\left(\omega-\omega(-\boldsymbol{k}_1^2)-\omega(-(\boldsymbol{k}_1-\boldsymbol{q})^2)\right)=$$
$$=\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)-\frac{2g^2}{\pi}\int d^2\boldsymbol{k}'\frac{F(\omega,\,\boldsymbol{k}',\,\boldsymbol{k}_2,\,\boldsymbol{q})}{\boldsymbol{k}_1^2(\boldsymbol{k}'^2-\boldsymbol{q})^2}\left(\boldsymbol{q}^2-\frac{\boldsymbol{k}_1^2(\boldsymbol{k}'-\boldsymbol{q})^2+\boldsymbol{k}'^2(\boldsymbol{k}_1-\boldsymbol{q})^2}{(\boldsymbol{k}_1-\boldsymbol{k}')^2}\right)\ . \tag{5.1.57}$$

This is the BFKL equation. Even though it is not obvious from this representation, the equation is IR finite, and the divergences of the reggeized gluon trajectories cancel the divergences of the kernel. It can be rewritten in a form in which this is manifest, but we will find the form (5.1.57) more suited to our needs.

The equation is rather difficult to solve - even though it was presented in its final form in 1978 [10] it was not until 1986 that Lipatov [217] provided a general solution. There is a special case which is simpler to solve, though, which is the case $t = 0$ of zero momentum transfer, known as the forward limit, giving the intercept of the pomeron trajectory.

In general, the BFKL equation takes the form

$$\omega F(\omega,\,\boldsymbol{k}_1,\,\boldsymbol{k}_2,\,\boldsymbol{q})=\delta^2(\boldsymbol{k}_1-\boldsymbol{k}_2)+K\bullet F(\omega,\,\boldsymbol{k}_1,\,\boldsymbol{k}_2,\,\boldsymbol{q})\ . \tag{5.1.58}$$

In the forward limit, where $\boldsymbol{q} = 0$, this is solved relatively easily by searching for a complete set of eigenfunctions $\phi_i(\boldsymbol{k})$ of K, where i will turn out to be a continuous index, for which

$$K\bullet\phi_i(\boldsymbol{k})=\kappa_i\phi_i(\boldsymbol{k})\ , \tag{5.1.59}$$

and

$$\sum_i\phi_i(\boldsymbol{k}_1)\phi_i^*(\boldsymbol{k}_2)=\delta^2(\boldsymbol{k}_1-\boldsymbol{k}_2)\ . \tag{5.1.60}$$

The eigenfunctions only depend on one transverse momentum since in the BFKL equation $\boldsymbol{k}_2$ is only a spectator variable. Inserting this expansion into (5.1.58) then gives the solution as

$$F(\omega,\, \boldsymbol{k}_1,\, \boldsymbol{k}_2,\, 0) = \sum_i \frac{\phi_i(\boldsymbol{k}_1)\phi_i^*(\boldsymbol{k}_2)}{\omega - \kappa_i} \ . \tag{5.1.61}$$

The eigenvalues thus give the singularities of the amplitude in the ω-plane, and thereby the Regge poles.

Performing the eigenvalue analysis one finds that the eigenvalues are labeled by a discrete index n and a continuous index ν, and

$$F(\omega,\, \boldsymbol{k}_1,\, \boldsymbol{k}_2,\, \boldsymbol{0}) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\nu}{4\pi^2 k_1 k_2} \left(\frac{k_1^2}{k_2^2}\right)^{i\nu} \frac{e^{in\Delta\theta_{12}}}{\omega - 4g^2\chi_n(\nu)} \ , \tag{5.1.62}$$

where $\Delta\theta_{12}$ is the angle between the vectors $\perp k_1$ and $\boldsymbol{k}_2$, and

$$\chi_n(\nu) = 2\psi(1) - \psi((|n|+1)/2 + i\nu) - \psi((|n|+1)/2 - i\nu) \ , \tag{5.1.63}$$

with $\psi(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)}$ the digamma function, the logarithmic derivative of the Euler gamma function. This expression, $\chi_n(\nu)$ is often called simply the BFKL kernel. Due to the continuous index ν, the amplitude does not have isolated poles in the ω-plane, but rather a series of branch cuts (one for each value of n).

In QCD this problem is cured at Next to Leading Logarithmic Order [218], through the introduction of an infrared scale, via the running of the coupling constant (see discussion in [212]), since this will discretize the branch cuts. In $\mathcal{N}=4$ Super Yang Mills, however, it would seem that the branch cuts remain even at higher orders. One can still extract the leading high energy behavior, though. The leading s behaviour will be given by those singularities that have the largest real part. The eigenvalues (5.1.63) are real, and have the property that they are monotonically decreasing in $|n|$ and $|\nu|$. The dominant eigenvalues are therefore given by $|n| = 0$ and $\nu \approx 0$. For small ν, we have the expansion

$$\chi_0(\nu) = 4\log 2 - 14\zeta(3)\nu^2 + \mathcal{O}(\nu^3) \ , \tag{5.1.64}$$

implying that the leading s behavior of the color singlet exchange amplitude for $t = 0$ would take the form

$$\mathcal{A} \sim \left(\frac{s}{Q^2}\right)^{\alpha(0)} \ , \tag{5.1.65}$$

where the intercept is given by

$$\alpha(0) = 1 + 16g^2 \log 2 \ . \tag{5.1.66}$$

Now, in the general case, where $\boldsymbol{q} \neq 0$, it is much more difficult to identify the eigenfunctions of the Kernel. We will not go into details, but Lipatov solved this problem

[217] by noting that the BFKL equation had a two-dimensional conformal symmetry, an $SL(2,C)$ symmetry, which is uncovered by taking the Fourier transform with respect to the transverse momenta, expressing the amplitude in impact-parameter space. This symmetry is a central part of the algebraic analysis that we will perform later in section 5.3, where we will conjecture the existence of a second, dual $SL(2,C)$.

Having identified the $SL(2,C)$ symmetry, the form of the eigenfunctions can then be determined using representation theory. In doing so, it is convenient to represent the two-dimensional impact parameter vectors $(x_k,\, y_k)$ in terms of complex numbers $\rho_k = x_k + i y_k$. Apart from two impact parameters ρ_1 and ρ_2 that are conjugate to $\boldsymbol{k}_1$, and $\boldsymbol{k}_1 - \boldsymbol{q}$, respectively, the eigenfunctions depend on the discrete index n and the continuous index ν as before. In addition there is a new continuous, impact parameter index ρ_0. The eigenfunctions then take the form

$$\phi_n^\nu(\rho_{10},\, \rho_{20}) = \left(\frac{\rho_{12}}{\rho_{10}\,\rho_{20}}\right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\,\rho_{20}^*}\right)^{\tilde{m}} , \tag{5.1.67}$$

where $m = \frac{1}{2} + i\nu + n/2$ and $\tilde{m} = \frac{1}{2} + i\nu - n/2$ are the values of the Casimirs of the conformal group, and where $\rho_{ij} \equiv \rho_i - \rho_j$. The arguments of the eigenfunctions reflect the fact that their dependence on ρ_1, ρ_2 and ρ_0 enter in precisely the combinations ρ_{10} and ρ_{20}.

Inserting the eigenfunctions into the kernel, one finds that the BFKL equation has a very curious property. The eigenvalue corresponding to (5.1.67) is simply given by the characteristic function (5.1.63). If the eigenfunctions satisfied a completeness relation, such as (5.1.60), the amplitude would then be given by an equation similar to (5.1.61). This is the case, although the completeness relation, which we will not show here, turns out to be slightly more complicated. Introducing ρ_1' as the complex number corresponding to the Fourier transform of $\boldsymbol{k}_2$, and ρ_2' as the transform of $\boldsymbol{k}_2 - \boldsymbol{q}$ [5], the Fourier transformed amplitude becomes

$$\tilde{F}(\omega,\, \rho_1,\, \rho_2,\, \rho_1',\, \rho_2') = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu \int d^2\rho_0 C(\nu,\, |n|) \frac{\phi_n^\nu(\rho_{10},\, \rho_{20}) \phi_n^{*\,\nu}(\rho_{1'0},\, \rho_{2'0})}{\omega - 4g^2 \chi_n(\nu)} , \tag{5.1.68}$$

where

$$C(\nu,\, |n|) = \frac{16\nu^2 + 4n^2}{(4\nu^2 + 1 - n^2)^2 + 16 n^2 \nu^2} \tag{5.1.69}$$

is a factor arising from the form of the completeness relation. One could have absorbed this factor in the definition of the eigenfunctions, but that would have complicated their form instead.

This means that the eigenvalues of the BFKL kernel have no dependence on t or ρ_0, whatsoever! This is very odd since it implies that the ω-plane singularities do not depend

[5] This notation is chosen in order to coincide with the one used by Lipatov.

on $\boldsymbol{q}$, and the LLA Regge trajectory for the Pomeron is thus independent of t, and simply given by (5.1.66). For this reason the perturbative pomeron is called the "hard" pomeron, while the actually observed pomeron trajectory, for which the pomeron is called the "soft" pomeron, has a considerably softer t-dependence (implying a suppression at large $|t|$). Furthermore, the experimentally observed pomeron intercept is approximately 1.08, which for typical values of the strong coupling is much smaller than (5.1.66). The intercept is reduced when introducing NLLA [218] corrections, but then a problem is that the NLLA terms turn out to be larger (for typical values of the coupling) than the LLA ones, casting doubt on the validity of this approximation scheme. There are recent proposals (see for example [213]) where a subset of all higher order terms are resummed, by, for example, taking fully into account the scale dependence, and which produce an important part of the full amplitude in the Regge limit. Slightly different results are obtained depending on the convention chosen, however.

5.2 Integrability in the Regge limit - the Lipatov-Faddeev-Korchemsky spin chain

Putting aside the possibility that subleading corrections may be more important than the LLA for typical values of the coupling, there is a more fundamental problem. It turns out that the LLA does not satisfy the constraints imposed by unitarity. Most notably, the so-called Froissart-Martin bound [219] states that as $s \to \infty$, the total cross sections of a unitary theory cannot rise faster than $\log^2 s$. For hard pomeron exchange, however, we found that the total cross section increases as a positive power of s, meaning that the LLA does not describe a unitary theory of high energy scattering.

The bound puts a constraint on the allowed intercepts of Regge trajectories, implying that they must be less than or equal to unity. In the previous section it was mentioned that the experimental pomeron intercept was 1.08. This does not clash with unitarity, however, since it only means that at the energies available in the experiments, the pomeron provides the leading contribution. As one goes to even higher energies, other effects must enter that compensate for the pomeron intercept.

Returning to the perturbative picture, one would like to correct the LLA, by adding the diagrams necessary to recover unitarity, thereby obtaining the so-called generalized LLA. Bartels [223] worked out that the minimal subset of diagrams required for unitarity are the ones corresponding to the exchange of an arbitrary number L of reggeized gluons in the t-channel, but where this number is conserved for each diagram[6]. As an example,

[6] There are also other approaches to make the theory unitary. One way is to work in the so-called dipole

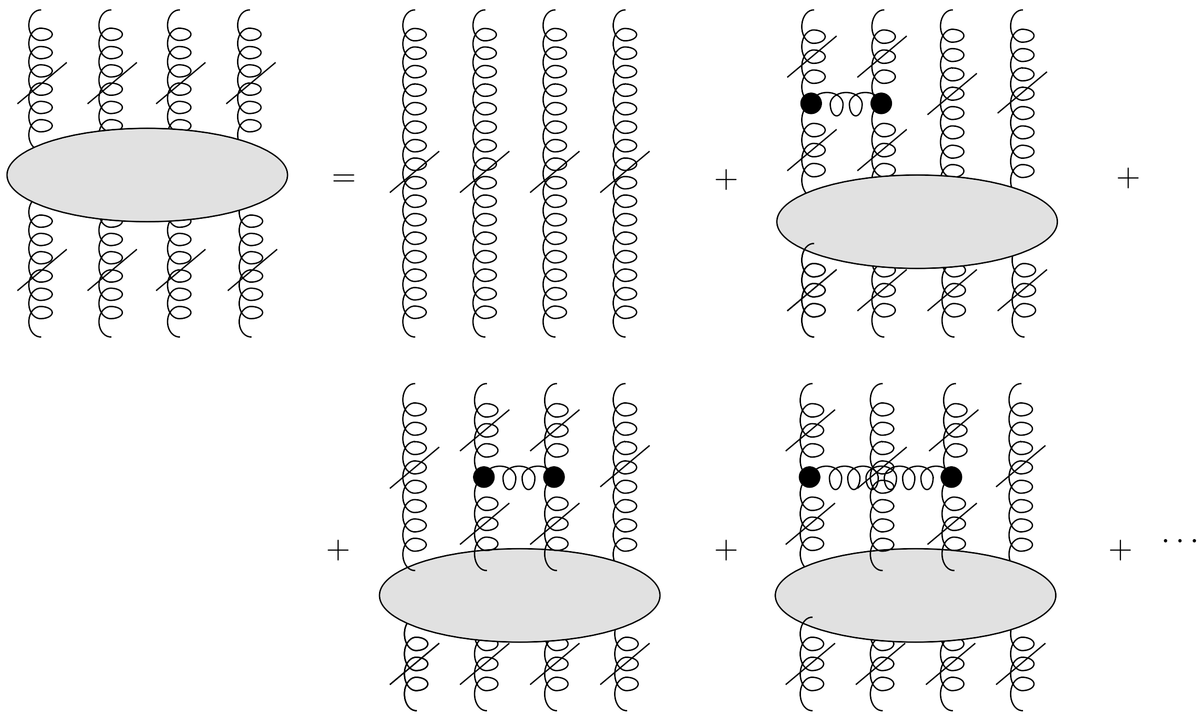

Figure 5.15: The equation determining the contribution to the generalized LLA coming from the exchange of 4 reggeized gluons. The dots indicate omitted terms featuring all possible combinations of pairwise contractions.

in figure 5.15 is shown the equation that determines the contribution from the exchange of 4 reggeized gluons. In this approximation one maintains the Regge kinematics and the conformal invariance of the BFKL equation.

This type of diagrams was further analyzed in [224], and then Lipatov noticed [225] that in the planar limit, where $N \to \infty$, one is left with interactions only among neighboring gluons, since diagrams such as the lower right one of figure 5.15 are subleading, and it is then natural to represent the action of the integration kernel on an L-reggeon state as the action of a nearest-neighbor Hamiltonian on a spin chain of L sites. Faddeev and Korchemsky then showed [16] that the Hamiltonian involved actually gives an integrable spin chain, known as the Heisenberg $XXX_{s=0}$ model, with representation of non-compact spin $s = 0$ at each site. The Hamiltonian for this model is of the form

$$H = \sum_{i=1}^{L} H_{i,\,i+1} \ , \tag{5.2.1}$$

theory [220], where the BFKL equation is obtained as an evolution equation for color dipole densities by taking into consideration the scattering of a isolated pair of color dipoles. Unitarity is then achieved by allowing for multiple scatterings [221]. An alternate way is to construct an effective action for high-energy gauge theories [222], treating for example reggeized gluons as particles of the Lagrangian.

with $H_{i,\,i+1}$ acting on sites i and $i+1$, and periodic boundary conditions $H_{L,\,L+1} = H_{L,\,1}$. More precisely

$$H_{i,\,i+1} = 2\psi(1) - \psi(J_{i,\,i+1} + 1) - \psi(-J_{i,\,i+1}) \ , \tag{5.2.2}$$

where $J_{i,\,i+1}$ is determined by the operator equation

$$J_{i,\,i+1}(J_{i,\,i+1} + 1) = 2S_i \otimes S_{i+1} \ , \tag{5.2.3}$$

where S_i is a spin 0 representation acting on site i, obtained by writing down a representation for general complex spin s and taking the limit $s \to 0$.

In order to be able to write down an analytic expression for the full singlet exchange amplitude in generalized LLA one must sum over contributions from arbitrarily large L. For non-zero s, the Heisenberg XXX_s model admits diagonalization by the Bethe Ansatz technique, such as we saw was the case for the spin-1/2 one-loop dilatation operator in the $SU(2)$-sector. The case $s = 0$ is degenerate, however, making it a priori unclear how to apply the Bethe Ansatz. Faddeev and Korchemsky were able to map the problem of diagonalizing the XXX_0-chain into a diagonalization of the XXX_{-1} chain, which does admit a Bethe Ansatz. This generalized Bethe Ansatz was then developed further in [226].

5.3 A dual conformal symmetry of BFKL

In the previous section we saw that one finds an integrable spin chain in the generalized LLA of high energy gauge theories. From the viewpoint of QCD the emergence of this infinite amount of symmetry in the high energy limit seems very mysterious since the Lagrangian has only a finite-dimensional symmetry algebra. However, in chapter 4 we saw that the equality of tree-level gluon scattering amplitudes for gauge theories explained their simplicity. As an example, the Yangian symmetry of $\mathcal{N} = 4$ Super Yang Mills fixes the MHV amplitudes to take the Parke-Taylor form (4.1.10). In the same way, the generalized LLA of gauge theories having the same gauge group coincides, since the only Feynman diagrams relevant in the Regge limit are those composed solely of gluons, providing a possible explanation for its integrability. Since planar $\mathcal{N} = 4$ is believed to be integrable, there should exist an exact solution for its scattering amplitudes, which will then induce a solution in the Regge limit. A further indication of this relation is that the integrability of the generalized LLA only appears in the planar limit, just as is the case of the integrability of the maximally supersymmetric theory.

To date, however, it is not understood how the integrable structure of $\mathcal{N} = 4$ implies the symmetries of BFKL and its extensions. Part of the difficulty lies in that the Regge limit is most naturally understood in terms of a two-dimensional effective theory living in the transverse plane (see for example [222]), while the symmetries of the supersymmetric gauge theory act in four dimensions. Another, trickier problem is of course that the problem of solving scattering amplitudes in $\mathcal{N} = 4$ has not been completed, so we do not know what the integrable structure is in the first place. What is known, however, is that at tree level, and probably at higher orders, $\mathcal{N} = 4$ scattering amplitudes exhibit the Yangian symmetry discussed in section 4.4. There, we saw that the Yangian symmetry was generated by the ordinary and dual superconformal symmetries.

In the constructions of Lipatov, Faddeev and Korchemsky, the conformal invariance of the BFKL equation is very important. Without it, one cannot expect to obtain a simple and integrable spin chain Hamiltonian, such as the Heisenberg model. Indeed, at each site of their spin chain sits a non-compact representation of the $SL(2, C)$-symmetry. It could be argued that this conformal symmetry comes from the symmetry of the QCD Lagrangian, since it is classically conformally invariant, and the consequences of a non-vanishing beta-function do not enter at LLA. However, if one constructs NLLA BFKL in $\mathcal{N} = 4$ one finds, as was settled only recently [227], that the $SL(2, C)$ symmetry remains at next-to-leading order, while the same is not true for QCD. It is therefore natural to view the $SL(2, C)$-symmetry as being what is left of the conformal symmetry of $\mathcal{N} = 4$ after taking the Regge limit.

The natural question that then arrises is if there is anything left of the dual conformal symmetry. If this where the case, perhaps the integrability of the Lipatov-Faddeev-Korchemsky spin chain could be understood as being generated by these two finite symmetry algebras. The topic of this section, based on [228] and [229], is precisely the identification of a new $SL(2, C)$ symmetry of the BFKL equation. We will discuss how similar the symmetry is to its $\mathcal{N} = 4$ analogue, and see if it has the potential to explain the integrability of BFKL.

5.3.1 The dual $SL(2, C)$

In section 4.3 we explained that the dual conformal symmetry of $\mathcal{N} = 4$ was uncovered by introducing a new set of kinematic variables x_i^μ, related to the incoming momenta p_i^μ of the scattering amplitude via

$$p_i = x_i - x_{i+1} \equiv x_{i\,i+1} \; . \tag{5.3.1}$$

This made the one-loop integral take the form

$$\int \frac{d^4 x_I \, x_{13}^2 x_{24}^2}{x_{1I}^2 x_{2I}^2 x_{3I}^2 x_{4I}^2} \; . \tag{5.3.2}$$

which is formally invariant under a dual conformal symmetry acting on the x_i^μ. At the same time, we saw in the previous section that, under the assumption of reggeization, the reggeized gluon trajectory could be obtained from a one-loop calculation. There, we used the Cutkosky rules to obtain the imaginary part of the amplitude, proportional to the trajectory, by putting a pair of propagators on-shell. Doing so corresponds to setting the upper and lower propagators of 4.2 on-shell, removing the propagators x_{1I}^2 and x_{3I}^2 from the denominator of (4.3.4) and introducing two delta functions, thereby reducing the dimensions of the integral from four to two. Furthermore, in the Regge limit, the integral is dominated by the transverse components of the momenta, replacing the x by two-dimensional vectors $\boldsymbol{x}$. Taking into consideration the tree-level factors (which corrects the numerator of (4.3.4)), one then obtains the reggeized gluon trajectory as

$$\omega(\boldsymbol{x}_{24}^2) = -\frac{g^2}{\pi} \int d^2\boldsymbol{x}_I \frac{\boldsymbol{x}_{24}^2}{\boldsymbol{x}_{2I}^2 \boldsymbol{x}_{4I}^2} \ . \tag{5.3.3}$$

We can obtain the same representation of the trajectory by introducing a new integration variable $\boldsymbol{x}_I$ in (5.1.22) through $\boldsymbol{k}' = \boldsymbol{x}_{2I}$, and performing the replacement $\boldsymbol{q} = \boldsymbol{p}_2 + \boldsymbol{p}_3 \to \boldsymbol{x}_{24}$. This last change is precisely what we would expect from a two-dimensional version of (5.3.1). Furthermore, the integral representation (5.3.3) has a formal two-dimensional inversion symmetry

$$\boldsymbol{x}_i \to \frac{\boldsymbol{x}_i}{\boldsymbol{x}^2} \tag{5.3.4}$$

since the squared differences x_{ij}^2 and the measure transforms, similarly to (4.3.5), as

$$\boldsymbol{x}_{ij}^2 \to \frac{\boldsymbol{x}_{ij}^2}{\boldsymbol{x}_i^2 \boldsymbol{x}_j^2} \ , \quad \text{and} \quad d^2\boldsymbol{x}_I \to \frac{d^2\boldsymbol{x}_I}{\boldsymbol{x}_I^4} \ . \tag{5.3.5}$$

This suggests that we look for a remnant of the dual conformal symmetry in the two-dimensional theory that describes the Regge limit by introducing $\boldsymbol{x}$-variables, related to the incoming, transverse momenta $\boldsymbol{p}_i$ by

$$\boldsymbol{p}_i = \boldsymbol{x}_i - \boldsymbol{x}_{i+1} \ , \tag{5.3.6}$$

and then acting on the $\boldsymbol{x}$-variables in the standard way. Together with the inversions, one also has translation symmetry (since the $\boldsymbol{x}$ only enter as the differences (5.3.6)) as well as rotation and dilatation symmetry (where all the x are multiplied by a constant). In total, this constitutes the group $SL(2, C)$. The dilatations and rotations coincide with the original $SL(2, C)$-symmetry (reminiscent of the relationship between the conformal and dual conformal symmetries of $\mathcal{N} = 4$), while translations and inversions will be different.

Since our main motivation for searching for a dual $SL(2,\, C)$ is to try to understand the integrability of the generalized LLA spin chain, it is natural to look at the BFKL equation next, considering that the BFKL Hamiltonian defines the two-site interaction in the spin chain. As above, we can write the BFKL equation as

$$\omega F(\omega,\, \boldsymbol{k}_A,\, \boldsymbol{k}_B,\, \boldsymbol{q}) = \delta^{(2)}(\boldsymbol{k}_A - \boldsymbol{k}_B) + \int d^2\boldsymbol{k}' K(\boldsymbol{k}_A,\, \boldsymbol{k}_A - \boldsymbol{q};\, \boldsymbol{k}',\, \boldsymbol{k}' - \boldsymbol{q}) F(\omega,\, \boldsymbol{k}',\, \boldsymbol{k}_B,\, \boldsymbol{q}) \tag{5.3.7}$$

where the kernel $K(\boldsymbol{k}_A,\, \boldsymbol{k}_A - \boldsymbol{q};\, \boldsymbol{k}',\, \boldsymbol{k}' - \boldsymbol{q})$ is given by

$$\frac{K_R(\boldsymbol{k}_A,\, \boldsymbol{k}_A - \boldsymbol{q}; -\boldsymbol{k}' + \boldsymbol{q},\, -\boldsymbol{k}')}{8\pi^3 \boldsymbol{k}_A^2 (\boldsymbol{k}' - \boldsymbol{q})^2} + \left[\omega(\boldsymbol{k}_A^2) + \omega((\boldsymbol{k}_A - \boldsymbol{q})^2)\right] \delta^{(2)}(\boldsymbol{k}_A - \boldsymbol{k}') \ , \tag{5.3.8}$$

with

$$K_R(\boldsymbol{p}_1,\, \boldsymbol{p}_2;\, \boldsymbol{p}_3,\, \boldsymbol{p}_4) = -N_c g_{YM}^2 \left[(\boldsymbol{p_3} + \boldsymbol{p_4})^2 - \frac{\boldsymbol{p}_2^2 \boldsymbol{p}_4^2}{(\boldsymbol{p}_2 + \boldsymbol{p}_3)^2} - \frac{\boldsymbol{p}_1^2 \boldsymbol{p}_3^2}{(\boldsymbol{p}_1 + \boldsymbol{p}_4)^2}\right] . \tag{5.3.9}$$

where $p_1,\, \ldots,\, p_4$ are incoming momenta.

Let us now rewrite equation (5.3.7) in terms of dual variables. Taken as incoming, the external momenta are $\boldsymbol{k}_A$, $-\boldsymbol{k}_A + \boldsymbol{q}$, $\boldsymbol{k}_B - \boldsymbol{q}$ and $-\boldsymbol{k}_B$ so the $\boldsymbol{x}$ variables are introduced as

$$\boldsymbol{p}_1 = \boldsymbol{x}_{12} = \boldsymbol{k}_A,\, \boldsymbol{p}_2 = \boldsymbol{x}_{23} = \boldsymbol{q} - \boldsymbol{k}_A,\, \boldsymbol{p}_3 = \boldsymbol{x}_{34} = \boldsymbol{k}_B - \boldsymbol{q},\, \boldsymbol{p}_4 = \boldsymbol{x}_{41} = -\boldsymbol{k}_B. \tag{5.3.10}$$

Solving for the original momenta we find $\boldsymbol{k}_A = \boldsymbol{x}_{12}$, $\boldsymbol{k}_B = \boldsymbol{x}_{14}$ and $\boldsymbol{q} = \boldsymbol{x}_{13}$, so $\boldsymbol{x}_1$ can also be interpreted as a shift of the origin for these momenta. Rewriting the kernel (5.3.8) in terms of the $\boldsymbol{x}_i$, with a change of integration variable through $\boldsymbol{k}' = \boldsymbol{x}_{1I}$, we get

$$K(\boldsymbol{x}_{12},\, \boldsymbol{x}_{32};\, \boldsymbol{x}_{1I},\, \boldsymbol{x}_{3I}) = \frac{K_R(\boldsymbol{x}_{12},\, \boldsymbol{x}_{23}; \boldsymbol{x}_{3I},\, \boldsymbol{x}_{I1})}{8\pi^3 \boldsymbol{x}_{12}^2 \boldsymbol{x}_{I3}^2} + \left[\omega(\boldsymbol{x}_{12}^2) + \omega(\boldsymbol{x}_{23}^2)\right] \delta^{(2)}(\boldsymbol{x}_{2I}) \ , \tag{5.3.11}$$

where

$$K_R(\boldsymbol{x}_{12},\, \boldsymbol{x}_{23}; \boldsymbol{x}_{3I},\, \boldsymbol{x}_{I1}) = -N_c g_{YM}^2 \left[\boldsymbol{x}_{13}^2 - \frac{\boldsymbol{x}_{23}^2 \boldsymbol{x}_{I1}^2}{\boldsymbol{x}_{2I}^2} - \frac{\boldsymbol{x}_{12}^2 \boldsymbol{x}_{I3}^2}{\boldsymbol{x}_{2I}^2}\right] \ . \tag{5.3.12}$$

Using that $\delta^{(2)}(\boldsymbol{x}_{2I}) \rightarrow \boldsymbol{x}_2^2 \boldsymbol{x}_I^2 \delta^{(2)}(\boldsymbol{x}_{2I})$ under conformal inversions, and that the Regge trajectories are formally invariant under the symmetry, one finds immediately that the kernel transforms as

$$K(\boldsymbol{x}_{12},\, \boldsymbol{x}_{32};\, \boldsymbol{x}_{1I},\, \boldsymbol{x}_{3I}) \rightarrow \boldsymbol{x}_2^2 \boldsymbol{x}_I^2 K(\boldsymbol{x}_{12},\, \boldsymbol{x}_{32};\, \boldsymbol{x}_{1I},\, \boldsymbol{x}_{3I}) \ . \tag{5.3.13}$$

We can then introduce this transformation in the BFKL equation. Using that the integration measure transforms according to (5.3.5), we find that a factor of $\frac{\boldsymbol{x}_2^2}{\boldsymbol{x}_I^2}$ is produced inside the integral. Now, if the Green function $F(\omega,\, \boldsymbol{x}_{12},\, \boldsymbol{x}_{14},\, \boldsymbol{x}_{13})$ were to produce a factor of $\boldsymbol{x}_2^2$ upon inversion, the factors of $\boldsymbol{x}_I^2$ would cancel in the integral and $K \otimes F$ would transform in the same way as F itself. Now, at lowest order in the coupling, F is simply given by the delta function, which indeed transforms in this way,

$$\delta^{(2)}(\boldsymbol{k}_A - \boldsymbol{k}_B) = \delta^{(2)}(\boldsymbol{x}_{24}) \rightarrow \boldsymbol{x}_2^2 \boldsymbol{x}_4^2 \delta^{(2)}(\boldsymbol{x}_{24}) \ . \tag{5.3.14}$$

Since F can be constructed through iterated application of the kernel, it follows that it should have the same conformal properties as the delta function. As a consequence, both the left and right hand sides of the BFKL equation transform in the same way under dual $SL(2, C)$ transformations, and so the equation is invariant. Let us also note that the way that F was introduced was purely conventional, and if we instead take $F' = \boldsymbol{x}_{24}^2 F$, and write the BFKL equation in terms of F', both F' and the kernel will be formally invariant.

5.3.2 IR divergences and anomalous ward identities

In $\mathcal{N} = 4$ SYM the dual conformal symmetry is broken by infrared divergences. As we mentioned in the previous section, BFKL is infrared finite, and the canceling of the divergences opens the possibility that the dual $SL(2, C)$-symmetry in its original form remain exact. However, this is unfortunately not the case. Perhaps the simplest way to see this is by studying the forward case. In general, we can write

$$\omega F = F_1(\hat{g}^2)\delta^{(2)}(\boldsymbol{k}_A - \boldsymbol{k}_B) + \frac{1}{(\boldsymbol{k}_A - \boldsymbol{k}_B)^2} F_2(\hat{g}^2) \; , \tag{5.3.15}$$

where F_1 and F_2 are functions of the momenta, the coupling and ω, entering through the combination $\hat{g}^2 \equiv \frac{g^2}{\omega}$. When $\boldsymbol{q} = 0$, $\boldsymbol{x}_1 = \boldsymbol{x}_3$, and since there is no way to form an $SL(2, C)$ invariant from only three coordinates, if F has the transformation properties of the delta function, F_1 and F_2 must be dual conformally invariant. This follows from the fact that $(\boldsymbol{k}_A - \boldsymbol{k}_B)^{-2}$ is the only function of the $\boldsymbol{x}$ that transforms correctly. F_2 can thus only be a function of $\hat{g}^2$. But when forming physical quantities one integrates over $\boldsymbol{k}_A$ and $\boldsymbol{k}_B$ and the divergences at $\boldsymbol{k}_A = \boldsymbol{k}_B$ must cancel between F_1 and F_2. The factor $(\boldsymbol{k}_A - \boldsymbol{k}_B)^{-2}$ is singular enough to cancel one factor of the trajectory, but F_1 is obtained by repeated application of the trajectory part of the kernel so, starting from the second iteration, products of two or more trajectories will appear and the divergences will fail to cancel.

A more direct way of observing the breaking of the symmetry is simply to calculate F_2, by regularizing the integrals and canceling the divergences order by order. The lowest order of F_2 is trivial to calculate, simply being the result of applying the kernel (5.3.11) to the inhomogenous delta-function term of (5.1.57). At the next order, one finds integrals of the form appearing in the trajectory. The gluon Regge trajectory is given, in dimensional regularization, by

$$\omega(\boldsymbol{x}_{12}^2) = -\frac{g^2}{\pi}(4\pi\mu)^{2\epsilon} \int d^{2-2\epsilon}\boldsymbol{x}_I \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{I1}^2 \boldsymbol{x}_{I2}^2} \approx -2g^2(4\pi e^{-\gamma})^{\epsilon} \left(\log \frac{\boldsymbol{x}_{12}^2}{\mu^2} - \frac{1}{\epsilon} \right) \; . \tag{5.3.16}$$

One also has to evaluate non-trivial integrals of the form

$$\frac{1}{\pi} \int \frac{d^2\boldsymbol{x}_I \, \boldsymbol{x}_{1I}^2}{\boldsymbol{x}_{2I}^2 \boldsymbol{x}_{3I}^2 \boldsymbol{x}_{4I}^2} \; , \tag{5.3.17}$$

but as shown in section 5.3.5 their form can be considerably restricted by studying their symmetry properties under dual $SL(2, C)$ transformations. In the end the poles in ϵ and the logarithms in the scale μ^2 cancel, as they must, but factors such as $\log \boldsymbol{x}_{12}^2$ do not and rather add up, producing an anomalous, non-invariant expression. One finds that

$$F_2(\hat{g}^2) = \frac{2\hat{g}^2}{\pi}(1+u-v) + \frac{8\hat{g}^4}{\pi}\left[(1+u-v)\log\left(\frac{\boldsymbol{x}_{24}^4}{\boldsymbol{x}_{12}^2\boldsymbol{x}_{34}^2}\right) + v\log v - u\log u\right] + \cdots \ , \tag{5.3.18}$$

where

$$u = \frac{\boldsymbol{x}_{14}^2\boldsymbol{x}_{23}^2}{\boldsymbol{x}_{12}^2\boldsymbol{x}_{34}^2} \quad \text{and} \quad v = \frac{\boldsymbol{x}_{13}^2\boldsymbol{x}_{24}^2}{\boldsymbol{x}_{12}^2\boldsymbol{x}_{34}^2} \tag{5.3.19}$$

are the two independent conformal invariants that can be formed from $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_4$. We see that the lowest order is invariant, while the next-to-lowest order, where the infra-red divergences make their entrance breaks the symmetry, due to an anomalous logarithmic term.

The appearance of these anomalous logarithms does not mean that the dual $SL(2, C)$ symmetry is broken beyond repair. We mentioned in section 4.3 that the scattering amplitudes satisfied the anomalous Ward identity (4.3.6), which one could correct for in such a way that the remainder of the amplitude had to be dual conformally invariant. Such a simple structure is not found in the case of the dual $SL(2, C)$, however. To the order calculated in (5.3.18), one has

$$iK_\mu^{(D)}\log F_2 = 2\gamma_1(\hat{g}^2)\left(x_{1\mu} + x_{3\mu} - x_{2\mu} - x_{4\mu}\right) \ , \tag{5.3.20}$$

where the generator $K_\mu^{(D)}$ of the dual $SL(2, C)$ is defined in the next section, μ is a two-dimensional index, $\gamma_1(\hat{\alpha}) = 4\hat{g}^2 + \mathcal{O}(\hat{\alpha}^2)$, and where we omit the bold font from the $\boldsymbol{x}$ when writing them out in components. But this does not hold to higher orders. We can see this by calculating one more order in the forward limit (which is tricky, but feasible). When $\boldsymbol{x}_1 = \boldsymbol{x}_3$, the solution of (5.3.20) is

$$\log F_2 = \gamma_1(\hat{g}^2)\log\left(\frac{\boldsymbol{x}_{24}^4}{\boldsymbol{x}_{12}^2\boldsymbol{x}_{14}^2}\right) + \gamma_2(\hat{g}^2) \ , \tag{5.3.21}$$

where γ_1 and γ_2 only depend on $\hat{g}^2$. Instead, one obtains

$$F_{2,\text{forward}}(\hat{g}^2) = 4\frac{\hat{g}^2}{\pi} + 16\frac{\hat{g}^4}{\pi}\log\left(\frac{\boldsymbol{x}_{24}^4}{\boldsymbol{x}_{12}^2\boldsymbol{x}_{14}^2}\right) + 64\frac{\hat{g}^6}{\pi}\left[\log^2\left(\frac{\boldsymbol{x}_{24}^4}{\boldsymbol{x}_{12}^2\boldsymbol{x}_{14}^2}\right) - \frac{1}{4}\log^2\frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{14}^2} - \frac{\pi^2}{3}\right] + \cdots \ . \tag{5.3.22}$$

The lack of a simple exponentiation of the breaking of the symmetry is unfortunate, but it does not mean that the symmetry can not be recovered. In the next section we will show that at least the leading order term (5.3.20) can be corrected by deforming the representation of the dual generators while preserving the commutation relations between the original and dual algebras. This is a strong indication that the dual $SL(2, C)$ may be part of an all-order symmetry.

5.3.3 The algebra of higher charges

We will now study what is obtained when the original and dual $SL(2,\,C)$ algebras are taken together, with the case of $\mathcal{N}=4$ Super Yang Mills in mind, in which the original and dual superconformal symmetries generated a Yangian algebra, as we discussed in section 4.4. We will therefore start by writing the two algebras in a common language, after which we will see if we can extract a generator from the dual $SL(2,\,C)$ taking the bi-local form (4.4.7) in terms of the original algebra.

Let us first review the action of the original $SL(2,C)$ symmetry of the BFKL Hamiltonian, discovered by Lipatov in [217], whose generators we will denote by $J_a^{(0)}$, in analogy with the level 0 generators of the $\mathcal{N}=4$ case. The symmetry is uncovered by performing a Fourier transform of F into impact parameter space:

$$\tilde{f}(\boldsymbol{\rho}) = \int d^2\boldsymbol{k}_A d^2\boldsymbol{k}_B d^2\boldsymbol{q} e^{i(\boldsymbol{\rho}_1\cdot\boldsymbol{k}_A+\boldsymbol{\rho}_2\cdot(\boldsymbol{q}-\boldsymbol{k}_A)-\boldsymbol{\rho}_3\cdot(\boldsymbol{q}-\boldsymbol{k}_B)-\boldsymbol{\rho}_4\cdot\boldsymbol{k}_B)}\frac{F(\boldsymbol{k}_A,\,\boldsymbol{k}_B,\,\boldsymbol{q})}{\boldsymbol{k}_B^2(\boldsymbol{k}_A-\boldsymbol{q})^2}\;, \tag{5.3.23}$$

The additional factors $\boldsymbol{k}_B^2$ and $(\boldsymbol{k}_A-\boldsymbol{q})^2$, appearing in (5.3.23) correspond to propagators removed from F in the normalization of the BFKL equation we use. This can be rewritten in the form of an ordinary Fourier transform in terms of incoming momenta $\boldsymbol{p}_i$ as

$$\tilde{f}(\boldsymbol{\rho}) = \int \prod_i d^2\boldsymbol{p}_i e^{i\boldsymbol{\rho}_i\cdot\boldsymbol{p}_i} f(\boldsymbol{p})\;, \tag{5.3.24}$$

where

$$f(\boldsymbol{p}) = \frac{F(\boldsymbol{p})}{\boldsymbol{p}_4^2\boldsymbol{p}_2^2}\delta^{(2)}\left(\sum \boldsymbol{p}_i\right)\;. \tag{5.3.25}$$

Introducing the complex coordinate $\rho=\rho^x+i\rho^y$ one finds an invariance of $\tilde{f}(\rho)$ when

$$\rho \to \frac{a+b\rho}{c+d\rho}\;, \tag{5.3.26}$$

where a, b, c, and d are complex parameters satisfying $ad-bc=1$. The transformations (5.3.26) thus represent the group $SL(2,C)$. This group is generated by the transformations

$$\rho \to a+\rho\;, \tag{5.3.27}$$

corresponding to translations,

$$\rho \to b\rho\;, \tag{5.3.28}$$

corresponding to dilatations (when b is real) and rotations (when b is a phase),

$$\rho \to \frac{1}{\rho}\;, \tag{5.3.29}$$

which is a complex inversion. All of these transformations are standard two-dimensional conformal transformations. With the exception of the complex inversion they all have analogous transformations in the four-dimensional conformal group, acting on a four-vector x^μ. The reason that the inversion is different is that in four-dimensions it appears as

$$x^\mu \to \frac{x^\mu}{x^2} \tag{5.3.30}$$

which would correspond in the two-dimensional case to

$$\boldsymbol{\rho} \to \frac{\boldsymbol{\rho}}{\boldsymbol{\rho}^2} \ , \tag{5.3.31}$$

which is

$$\rho \to \frac{1}{\rho^*} \ , \tag{5.3.32}$$

written in complex notation. It is this last transformation that we will refer to as the two-dimensional inversion. It is also a symmetry of BFKL since the kernel is real, implying invariance under $\rho \to \rho^*$.

In order to better see the relation with the four-dimensional conformal group, where we can view the $SL(2,C)$ symmetry as the subgroup comprised of dilatations, and rotations, translations and special conformal transformations with indices taking values in the transverse plane, we will now drop the convenient complex notation and instead use two-dimensional vector notation. When we write ρ_i^μ, the superscript is a two-dimensional index taking the values x and y, while the subscript labels the particle, taking values $i = 1, \ldots, n$. In the case of BFKL, n is 4, while it may be higher when we study its extensions. We will now construct the infinitesimal generators corresponding to the translations, rotations, dilatations and special conformal transformations, following the convention that an infinitesimal generator J_a produces a finite transformation through $e^{i\xi J_a}$.

In impact parameter space the generators are given by the standard expressions. Infinitesimal two-dimensional translations, result in the usual expression

$$P_\mu = -i \sum_i \frac{\partial}{\partial \rho_i^\mu} \tag{5.3.33}$$

for the momentum operator, while an infinitesimal dilatation induced the change gives the generator for dilatations as

$$D = -i \sum_i \rho_i^\mu \frac{\partial}{\partial \rho_i^\mu} \ . \tag{5.3.34}$$

An infinitesimal (counterclockwise) rotation gives the generator of rotations as

$$R = -i\epsilon_{\mu\nu} \sum_i \rho_i^\mu \frac{\partial}{\partial \rho_{i\,\nu}} \ , \tag{5.3.35}$$

with $\epsilon_{12} = 1$. The special conformal transformations are defined by the usual ITI (Inversion-Translation-Inversion) transformation giving

$$K_\mu = -i \sum_i \left(\boldsymbol{\rho}_i^2 \frac{\partial}{\partial \rho_i^\mu} - 2\rho_\mu \rho^\nu \frac{\partial}{\partial \rho_i^\nu} \right) \ . \tag{5.3.36}$$

We will now Fourier transform the generators. The reason is that the dual $SL(2, C)$ is more naturally written in the momentum representation, and in order to be able to combine the two symmetries they must be expressed in a common language. The way we perform the transformation is simply to act with the generators on

$$\tilde{f}(\boldsymbol{\rho}) = \int \prod_i \frac{d\boldsymbol{p}_i}{(2\pi)^2} e^{i\boldsymbol{\rho}_i \cdot \boldsymbol{p}_i} f(\boldsymbol{p}) \tag{5.3.37}$$

and rewrite, by partial integration, this as an action on $f(\boldsymbol{p})$. For simplicity we will denote the Fourier transformed generators by the same symbol as before.

We obtain

$$D = i \sum_i \left(2 + p_{i\mu} \frac{\partial}{\partial p_{i\mu}} \right) \ , \tag{5.3.38}$$

where the constant 2 stems from being in two dimensions,

$$R = -i\epsilon_{\mu\nu} \sum_i p_i^\mu \frac{\partial}{\partial p_{i\,\nu}} \ , \tag{5.3.39}$$

and

$$K_\mu = \sum_i \left(4\frac{\partial}{\partial p_i^\mu} + 2p_i^\nu \frac{\partial}{\partial p_i^\nu} \frac{\partial}{\partial p_i^\mu} - p_{i\mu} \frac{\partial}{\partial p_i^\nu} \frac{\partial}{\partial p_{i\nu}} \right) \ . \tag{5.3.40}$$

Here the coefficient in front of $\frac{\partial}{\partial p_i^\mu}$ is twice the number of dimensions, and is therefore 4 in our case. And finally, the momentum will of course be

$$P_\mu = \sum_i p_{i\,\mu} \ . \tag{5.3.41}$$

The dual symmetry in the bi-local form

We will now show how a piece can be extracted from the dual $SL(2, C)$ symmetry that is (almost) of the form (4.4.7). As explained above, starting from the set of incoming momenta $\{\boldsymbol{p}_i\}$, $i = 1, \ldots, n$, the the dual $SL(2, C)$ symmetry is uncovered by performing the change of variables

$$\boldsymbol{x}_i - \boldsymbol{x}_{i+1} = \boldsymbol{p}_i \ . \tag{5.3.42}$$

Momentum conservation is automatically satisfied by identifying $\boldsymbol{x}_1$ with $\boldsymbol{x}_{n+1}$, but since we are interested in the algebraic structure that results from commuting the ordinary and dual symmetries, we will let, $\boldsymbol{x}_1$ and $\boldsymbol{x}_{n+1}$ be independent variables, and include a factor $\delta(\boldsymbol{x}_1 - \boldsymbol{x}_{n+1})$ in the Green's function to impose momentum conservation, just as was done in [187] in the case of $\mathcal{N} = 4$.

In terms of the $\boldsymbol{x}$-variables, the generators of the dual $SL(2, C)$ take the same form as the generators (5.3.33)-(5.3.36) do in terms of the ρ-variables. It should be noted, though, that just as for the $\mathcal{N} = 4$ scattering amplitudes, not all the dual generators are invariances. The translations and the rotations are so, while the dilatations and the special conformal transformations are covariances. One can easily make them them into invariances, however, by shifting them by a constant term. This must be done before they can be combined with the generators (5.3.33)-(5.3.36). First, let us rewrite the generators of the dual algebra in terms of momenta p.

The inverse of the change of variables (5.3.42) is

$$\boldsymbol{x}_i = \boldsymbol{x}_1 - \sum_{j=1}^{i-1} \boldsymbol{p}_j \ , \tag{5.3.43}$$

where we have kept $\boldsymbol{x}_1$ as an independent variable, together with the momenta. Requiring that the $\boldsymbol{x}_i$ all be independent, in the sense that

$$\frac{\partial x_i^\mu}{\partial x_j^\nu} = \delta_j^i \delta_\nu^\mu \tag{5.3.44}$$

then implies that the derivatives with respect to the x should be replaced, when going to the p variables, as

$$\frac{\partial}{\partial x_1^\mu} \rightarrow \frac{\partial}{\partial p_1^\mu} + \frac{\partial}{\partial x_1^\mu} \tag{5.3.45}$$

$$\frac{\partial}{\partial x_i^\mu} \rightarrow \frac{\partial}{\partial p_i^\mu} - \frac{\partial}{\partial p_{i-1}^\mu} \ , \ i = 2, \ldots, n \tag{5.3.46}$$

$$\frac{\partial}{\partial x_{n+1}^\mu} \rightarrow -\frac{\partial}{\partial p_n^\mu} \ . \tag{5.3.47}$$

Performing these substitutions, the dilatation operator becomes

$$\begin{aligned} D^{(D)} &= -i \sum_{i=1}^{n+1} \boldsymbol{x}_i \cdot \frac{\partial}{\partial \boldsymbol{x}_i} = \\ &= -i \left(\boldsymbol{x}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} + \boldsymbol{x}_1 \cdot \frac{\partial}{\partial \boldsymbol{p}_1} + (\boldsymbol{x}_1 - \boldsymbol{p}_1) \cdot \left(\frac{\partial}{\partial \boldsymbol{p}_2} - \frac{\partial}{\partial \boldsymbol{p}_1} \right) + \cdots \right. \\ &\qquad \left. \cdots - (\boldsymbol{x}_1 - \boldsymbol{p}_1 - \cdots - \boldsymbol{p}_n) \cdot \frac{\partial}{\partial \boldsymbol{p}_n} \right) = \\ &= -i \left(\boldsymbol{x}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} + \sum_{i=1}^{n} \boldsymbol{p}_i \cdot \frac{\partial}{\partial \boldsymbol{p}_i} \right) \ , \end{aligned} \tag{5.3.48}$$

where we have introduced the superscript (D) to distinguish this set of generators from that of the ordinary conformal symmetry. We see that when acting on an object independent of $\boldsymbol{x}_1$ (or, equivalently, which only depends on the momenta), this dilatation operator is the same, up to a change of sign and a shift by a constant, as the original dilatation operator (5.3.38).

We do not obtain anything new from translations and rotations, either. Up to terms containing derivatives with respect to x_1 the former are represented by the identity, while the latter give the same generator as the original symmetry (5.3.39). The only new symmetry comes from the special conformal generators, mimicking the structure appearing in the scattering amplitudes of $\mathcal{N}=4$ SYM, in the form of the original and dual conformal generators.

The special conformal generators take the form

$$iK_\mu^{(D)} = \boldsymbol{x}_1^2 \frac{\partial}{\partial x_1^\mu} - 2x_{1\mu}\left(\boldsymbol{x}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} + \sum_i \boldsymbol{p}_i \cdot \frac{\partial}{\partial \boldsymbol{p}_i}\right) - 2\sum_i x_1^\nu \left(p_{i\mu}\frac{\partial}{\partial p_i^\nu} - p_{i\nu}\frac{\partial}{\partial p_i^\mu}\right) + \\ + \sum_i \left(2p_{i\mu}\boldsymbol{p}_i \cdot \frac{\partial}{\partial \boldsymbol{p}_i} - \boldsymbol{p}_i^2 \frac{\partial}{\partial p_i^\mu}\right) + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(p_{i\mu}\boldsymbol{p}_j \cdot \frac{\partial}{\partial \boldsymbol{p}_i} + p_{j\mu}\boldsymbol{p}_i \cdot \frac{\partial}{\partial \boldsymbol{p}_i} - \boldsymbol{p}_i \cdot \boldsymbol{p}_j \frac{\partial}{\partial p_i^\mu}\right) . \tag{5.3.49}$$

Again, when acting on a physical object, which only depends on the momenta, all of the derivatives with respect to x_1 can be dropped. Furthermore, the third term can also be dropped due to the rotational invariance. The remaining x_1-dependence is given by

$$-2x_{1\mu}\sum_i \boldsymbol{p}_i \cdot \frac{\partial}{\partial \boldsymbol{p}_i} . \tag{5.3.50}$$

As mentioned above, $K_\mu^{(D)}$ is not an invariance of the gluon Green's function, and must be corrected by a constant piece. In order to do so we will hereafter restrict our attention to the case $n=4$ of BFKL for which we know the transformation properties. Under dual inversions $F \to \boldsymbol{x}_2^2\boldsymbol{x}_4^2 F$, and from (5.3.25) we get

$$f(\boldsymbol{p}) = \frac{F(\boldsymbol{x})}{\boldsymbol{x}_{14}^2\boldsymbol{x}_{23}^2}\delta^{(2)}(\boldsymbol{x}_{15}) \to \boldsymbol{x}_1^6\boldsymbol{x}_2^4\boldsymbol{x}_3^2\boldsymbol{x}_4^4 f(\boldsymbol{p}) . \tag{5.3.51}$$

This implies that acting on $f(\boldsymbol{p})$ with $iK_\mu^{(D)}$ produces the factor

$$6x_{1\mu} + 4x_{2\mu} + 2x_{3\mu} + 4x_{4\mu} = 16x_{1\mu} - 10p_{1\mu} - 6p_{2\mu} - 4p_{3\mu} , \tag{5.3.52}$$

and we see that the gluon Green's function will be invariant under the combination

$$iK_\mu^{(D)} + 10p_{1\mu} + 6p_{2\mu} + 4p_{3\mu} - 16x_{1\mu} . \tag{5.3.53}$$

Here, we see that the $-16x_{1\mu}$ term is precisely what is needed in order to eliminate the remaining $\boldsymbol{x}_1$ dependence completely. When added to (5.3.50), one obtains a term which is proportional to (5.3.38) and can be dropped. In the end, we extract the new generator

$$\hat{K}_\mu = \sum_i \left(2p_{i\mu}\boldsymbol{p}_i \cdot \frac{\partial}{\partial \boldsymbol{p}_i} - \boldsymbol{p}_i^2 \frac{\partial}{\partial p_i^\mu} \right) + 10p_{1\mu} + 6p_{2\mu} + 4p_{3\mu} +$$
$$+ 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} \left(p_{i\mu}\boldsymbol{p}_j \cdot \frac{\partial}{\partial \boldsymbol{p}_i} + p_{j\mu}\boldsymbol{p}_i \cdot \frac{\partial}{\partial \boldsymbol{p}_i} - \boldsymbol{p}_i \cdot \boldsymbol{p}_j \frac{\partial}{\partial p_i^\mu} \right) \tag{5.3.54}$$

from the dual $SL(2, C)$.

We will now go on to remove a part of (5.3.54), which itself annihilates the Green's function, so that what is left is almost of the bi-local form (4.4.7). The relevant bi-local operators turn out to be those where the index a corresponds to the two-dimensional momentum P_μ. Using the metric $g_{ab} = \frac{1}{4} f_{ac}{}^e f_{be}{}^c$ for the algebra to raise indices, where the constant 1/4 is chosen for convenience, and inserting the level 0 generators (5.3.38)-(5.3.41) into (4.4.7) we have the operator

$$\tilde{J}^{(1)}_\mu = -i \sum_{1 \le j < i \le n} \left[P_{j\mu} D_i - \epsilon_{\mu\nu} P_{j\nu} R_i - (i \leftrightarrow j) \right] =$$
$$= \sum_{1 \le j < i \le n} \left[p_{j\mu} \left(2 + p_i^\rho \frac{\partial}{\partial p_i^\rho} \right) + \epsilon_{\mu\nu} p_j^\nu \epsilon_{\rho\lambda} p_i^\rho \frac{\partial}{\partial p_{i\lambda}} - (i \leftrightarrow j) \right] , \tag{5.3.55}$$

where a tilde is added to indicate that we do not know yet if these are symmetries of the Green's function.

In order to arrive at (5.3.55) let us start by splitting the last sum in (5.3.54) into two equal pieces. One of the pieces we leave in it's current form, while using $P_\mu = \sum p_{i\mu}$ we rewrite the second piece as

$$\sum_{i=2}^{n} \left(p_{i\mu}(\boldsymbol{P} - \boldsymbol{p}_i - \cdots - \boldsymbol{p}_n) \cdot \frac{\partial}{\partial \boldsymbol{p}_i} + (P_\mu - p_{i\mu} - \cdots - p_{n\mu})\boldsymbol{p}_i \cdot \frac{\partial}{\partial \boldsymbol{p}_i} - \right.$$
$$\left. - \boldsymbol{p}_i \cdot (\boldsymbol{P} - \boldsymbol{p}_i - \cdots - \boldsymbol{p}_n) \frac{\partial}{\partial p_i^\mu} \right) . \tag{5.3.56}$$

The nested sums run from index i to n, and we can cancel the terms corresponding to index i with terms from the first line of (5.3.54). The only part that then remains from the sum in the first line is the term corresponding to $i = 1$,

$$2p_{1\mu}\boldsymbol{p}_1 \cdot \frac{\partial}{\partial \boldsymbol{p}_1} - \boldsymbol{p}_1^2 \frac{\partial}{\partial p_1^\mu} =$$
$$p_{1\mu}(\boldsymbol{P} - \boldsymbol{p}_2 - \cdots - \boldsymbol{p}_n) \cdot \frac{\partial}{\partial \boldsymbol{p}_1} + (P_\mu - p_{2\mu} - \cdots - p_{n\mu})\boldsymbol{p}_1 \cdot \frac{\partial}{\partial \boldsymbol{p}_1} -$$
$$- \boldsymbol{p}_1 \cdot (\boldsymbol{P} - \boldsymbol{p}_2 - \cdots - \boldsymbol{p}_n) \frac{\partial}{\partial p_1^\mu} . \tag{5.3.57}$$

These terms complete the sum in (5.3.56) so that it starts from $i = 1$. Altogether, we find

$$\begin{aligned}\hat{K}_\mu = &\sum_{i=1}^{n}\left(p_{i\mu}\boldsymbol{P}\cdot\frac{\partial}{\partial\boldsymbol{p}_i}+P_\mu\boldsymbol{p}_i\cdot\frac{\partial}{\partial\boldsymbol{p}_i}-\boldsymbol{p}_i\cdot\boldsymbol{P}\frac{\partial}{\partial p_i^\mu}\right)+10p_{1\mu}+6p_{2\mu}+4p_{3\mu}+\\&+\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left(p_{i\mu}\boldsymbol{p}_j\cdot\frac{\partial}{\partial\boldsymbol{p}_i}+p_{j\mu}\boldsymbol{p}_i\cdot\frac{\partial}{\partial\boldsymbol{p}_i}-\boldsymbol{p}_i\cdot\boldsymbol{p}_j\frac{\partial}{\partial p_i^\mu}\right)-\\&-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(p_{i\mu}\boldsymbol{p}_j\cdot\frac{\partial}{\partial\boldsymbol{p}_i}+p_{j\mu}\boldsymbol{p}_i\cdot\frac{\partial}{\partial\boldsymbol{p}_i}-\boldsymbol{p}_i\cdot\boldsymbol{p}_j\frac{\partial}{\partial p_i^\mu}\right)\ .\end{aligned}\tag{5.3.58}$$

The first sum will itself annihilate the Green's function. In fact, it will annihilate any function of the form $\delta^{(2)}(\boldsymbol{P})\,h(\{\boldsymbol{p}_i\})$, which includes a momentum conserving delta function. The factors of P make all terms in which the derivatives act on h vanish so only the terms in which the derivatives act on the delta function remain. For the same reason, when the derivatives are moved off the delta function by partial integration they must act on the factors of P in order to get a non-vanishing contribution. We are then left with

$$-\sum_{i=1}^{n}\left(p_{i\mu}2+p_{i\mu}-p_{i\mu}\right)\delta^{(2)}(\boldsymbol{P})\,h(\{\boldsymbol{p}_i\})\ ,\tag{5.3.59}$$

which once again vanishes due to the delta function.

Extracting the first term from (5.3.58), what remains is

$$\sum_{1\le j<i\le n}\left(p_{i\mu}\boldsymbol{p}_j\cdot\frac{\partial}{\partial\boldsymbol{p}_i}+p_{j\mu}\boldsymbol{p}_i\cdot\frac{\partial}{\partial\boldsymbol{p}_i}-\boldsymbol{p}_i\cdot\boldsymbol{p}_j\frac{\partial}{\partial p_i^\mu}-(i\leftrightarrow j)\right)+10p_{1\mu}+6p_{2\mu}+4p_{3\mu}\ ,\tag{5.3.60}$$

which can be rewritten (in two dimensions) as

$$\sum_{1\le j<i\le n}\left(p_{j\mu}p_i^\rho\frac{\partial}{\partial p_i^\rho}+\epsilon_{\mu\nu}p_j^\nu\epsilon_{\rho\sigma}p_i^\rho\frac{\partial}{\partial p_{i\sigma}}-(i\leftrightarrow j)\right)+10p_{1\mu}+6p_{2\mu}+4p_{3\mu}\ ,\tag{5.3.61}$$

reproducing a large portion of the terms in (5.3.55). The only one that remains unaccounted for is

$$\sum_{1\le j<i\le n}2(p_{j\mu}-p_{i\mu})=4\left((n-1)p_{1\mu}+(n-2)p_{2\mu}+\cdots+p_{n-1}-(n-1)P_\mu\right)\ .\tag{5.3.62}$$

For the case that interests us, which is $n = 4$, this becomes, after discarding the term proportional to P_μ,

$$12p_{1\mu}+8p_{2\mu}+4p_{3\mu}\ ,\tag{5.3.63}$$

which is almost, but not exactly, equal to the term $10p_{1\mu}+6p_{2\mu}+4p_{3\mu}$, present in (5.3.61). The difference is $2p_{1\mu}+2p_{2\mu}\equiv 2q_\mu$.

We must thus conclude that, in contrast to the case of $\mathcal{N}=4$ SYM, the bi-local operators $\tilde{J}^{(1)}_\mu$ are in general not symmetries of the Green's function. We instead have the symmetries

$$J^{(1)}_\mu \equiv \tilde{J}^{(1)}_\mu - 2q_\mu \ . \tag{5.3.64}$$

Commuting these generators with the level 0 algebra gives the rest of the "level 1" generators, all of which can be written as as

$$J^{(1)}_a \equiv \tilde{J}^{(1)}_a - 2J^{(0)}_{(12)a} \ , \tag{5.3.65}$$

where

$$J^{(0)}_{(12)a} \equiv \sum_{i=1}^{2} J^{(0)}_{ia} \tag{5.3.66}$$

are the level 0 generators restricted to the first two (upper) momenta, and where the $\tilde{J}^{(1)}_a$ are defined by the bi-local formula.

It should be noted, that the reason that we are forced to deform the $J^{(1)}_a$ generators is the non-symmetrical action (5.3.51) of the dual conformal inversions. The $\tilde{J}^{(1)}_a$ would have been symmetries if $F(p)$ had produced a factor $x_1^4x_2^4x_3^4x_4^4$ under inversions [7].

The full algebra and higher orders

So, does the original $SL(2,C)$ together with (5.3.65) generate a Yangian? It follows from the form of the bi-local formula that the Yangian commutation relations (4.4.2) are satisfied with the $\tilde{J}^{(1)}$, implying that they are also obeyed by the full $J^{(1)}$, since the additional $2J^{(0)}_{(12)a}$ piece simply gives the level 0 algebra restricted to particles 1 and 2. The key issue is therefore whether the Serre relations (4.4.3)-(4.4.4) are satisfied.

We mentioned in section 4.4 that for $PSU(2,2|4)$ the first Serre relation implies the second one. And in our case the first Serre relation is indeed satisfied. However, for $SL(2,C)$ it is trivially satisfied (its structure constants imply it take the form $0=0$), meaning that it is the second Serre relation that must be checked. Unfortunately, it turns out to not be satisfied. From the form of (4.4.3) and (4.4.4) it is clear that if the right hand side of the first relation is zero, the right hand side of the second relation must also be so. But, a quick check reveals that the left hand side becomes non-zero when using the generators (5.3.65). So it does not seem that the closure of the original and dual $SL(2,C)$-algebras give a Yangian algebra. This does not mean that the algebra cannot be infinite

[7] This should be compared with the amplitudes of $\mathcal{N}=4$ for which the inversion produces the factor $x_1^2\cdots x_n^2$, consistent with the bi-local formula because the relevant constants appearing in the generators (such as the constant in the dilatation operator) are half of what they are in the present case.

dimensional, though. The generators $\left\{J_a^{(0)},\, J_a^{(1)}\right\}$ do not close under commutation, and from their form it seems unlikely that their closure would be finite-dimensional.

An important issue is also whether the algebra itself is affected when the representation of the dual $SL(2,C)$ is deformed to $J_a^{(1)}(\hat{g}^2)$ in order to take into consideration the anomaly (5.3.20). We can see that the full form of the deformed generator becomes

$$J_a^{(1)}(\hat{g}^2) = \tilde{J}_a^{(1)} - 2J_{(12)a}^{(0)} - 8\hat{g}^2 J_{(13)a}^{(0)} + \mathcal{O}\left(\hat{g}^4\right) \ , \tag{5.3.67}$$

where $J_{(13)a}^{(0)}$ is defined analogously to $J_{(12)a}^{(0)}$. Interestingly, in the same way that the shift in (5.3.65) by $-2J_{(12)a}^{(0)}$ does not alter the commutation relation (4.4.2), neither does the change in (5.3.67). Also when considering the rest of the algebra, generated by $J_a^{(0)}$ and $J_a^{(1)}(\hat{g}^2)$ it does not seem that the commutation relations are altered by the introduction of the coupling dependence, meaning that the algebra obtained at lowest order seems to remain at next-to-leading order, and may very well be exact to all orders in the coupling. We will, however, leave the exact determination of this algebraic structure to future studies.

5.3.4 The $2 \to m$ reggeized gluon vertex

Apart from the interchange of several reggeized gluons, as was incorporated in the generalized LLA, a complete description of the Regge limit requires the inclusion of diagrams in which the number of reggeized gluons is not conserved in the t-channel. In particular, one includes a $2 \to m$ reggeized gluon vertex (see [230]), shown in figure 5.16, which is a generalization of the BFKL kernel K_R, and given by

$$\begin{aligned} &K_{2\to m}^{\{b\}\to\{a\}}(\boldsymbol{p}_2,\, \boldsymbol{p}_3;\, \boldsymbol{p}_4,\, \ldots,\, \boldsymbol{p}_{m+2},\, \boldsymbol{p}_1) = f_{a_1 b_1 c_1} f_{c_1 a_2 c_2} \cdots f_{c_{m-1} a_m b_2} g_{YM}^m \\ &\times \left[(\boldsymbol{p}_4 + \cdots + \boldsymbol{p}_1)^2 - \frac{\boldsymbol{p}_3^2(\boldsymbol{p}_5 + \cdots + \boldsymbol{p}_1)^2}{(\boldsymbol{p}_3 + \boldsymbol{p}_4)^2} - \frac{\boldsymbol{p}_2^2(\boldsymbol{p}_4 + \cdots + \boldsymbol{p}_{m+2})^2}{(\boldsymbol{p}_1 + \boldsymbol{p}_2)^2} + \frac{\boldsymbol{p}_1^2 \boldsymbol{p}_3^2(\boldsymbol{p}_5 + \cdots + \boldsymbol{p}_{m+2})^2}{(\boldsymbol{p}_1 + \boldsymbol{p}_2)^2(\boldsymbol{p}_3 + \boldsymbol{p}_4)^2}\right] \ , \end{aligned} \tag{5.3.68}$$

where the a_1, b_1 etc. are the color indices of the reggeized gluons and f_{ijk} the structure constants of the gauge group $SU(N)$.

Written in terms of $\boldsymbol{x}$ variables the vertex becomes

$$\begin{aligned} &K_{2\to m}^{\{b\}\to\{a\}}(\boldsymbol{x}_{23},\, \boldsymbol{x}_{34};\, \boldsymbol{x}_{45},\, \ldots,\, \boldsymbol{x}_{m+2\,1},\, \boldsymbol{x}_{12}) = \\ &f_{a_1 b_1 c_1} f_{c_1 a_2 c_2} \cdots f_{c_{m-1} a_m b_2} g_{YM}^m \left[\boldsymbol{x}_{24}^2 - \frac{\boldsymbol{x}_{34}^2 \boldsymbol{x}_{25}^2}{\boldsymbol{x}_{35}^2} - \frac{\boldsymbol{x}_{23}^2 \boldsymbol{x}_{14}^2}{\boldsymbol{x}_{13}^2} + \frac{\boldsymbol{x}_{23}^2 \boldsymbol{x}_{34}^2 \boldsymbol{x}_{15}^2}{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{35}^2}\right] \ , \end{aligned} \tag{5.3.69}$$

which we see is manifestly dual conformally covariant. We take this as an indication that the full effective high energy theory has, not only the original $SL(2,\, C)$-symmetry, as was

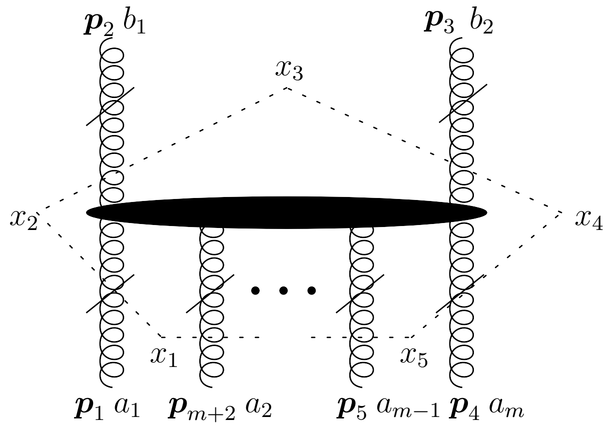

Figure 5.16: The $2 \to m$ reggeized gluon vertex. All momenta are taken as ingoing.

shown for the $2 \to 4$ vertex in [231], but also the dual $SL(2,\, C)$ symmetry. Furthermore, the BK equation [232], which is an extension of the BFKL equation in which one adds the $2 \to 4$ vertex, should have the dual $SL(2,\, C)$ symmetry as well.

5.3.5 Applying the $SL(2,\, C)$

Before ending our treatment of the dual $SL(2,\, C)$ let us note that it can be useful, not only for theoretical reasons, but also from a practical point of view. In particular, we will show how the form of the integral

$$I_1 \equiv \frac{1}{\pi} \int \frac{d^2\boldsymbol{x}_I\, \boldsymbol{x}_{1I}^2}{\boldsymbol{x}_{2I}^2 \boldsymbol{x}_{3I}^2 \boldsymbol{x}_{4I}^2} \tag{5.3.70}$$

can be restricted by using dual $SL(2, C)$ symmetry. First, we note that the integrand of I_1 has the same behavior close to the singularities at $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ and $\boldsymbol{x}_4$, and the same transformation properties under dual $SL(2, C)$ transformations as the integrand of

$$I_2 \equiv \frac{1}{2\pi} \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{23}^2 \boldsymbol{x}_{24}^2} \int d^2\boldsymbol{x}_I \left[(1+v-u) \frac{\boldsymbol{x}_{23}^2}{\boldsymbol{x}_{2I}^2 \boldsymbol{x}_{3I}^2} + (1-v+u) \frac{\boldsymbol{x}_{24}^2}{\boldsymbol{x}_{2I}^2 \boldsymbol{x}_{4I}^2} + (-1+v+u) \frac{\boldsymbol{x}_{34}^2}{\boldsymbol{x}_{3I}^2 \boldsymbol{x}_{4I}^2} \right] , \tag{5.3.71}$$

which can be evaluated directly in dimensional regularization using (5.3.16), where

$$u = \frac{\boldsymbol{x}_{14}^2 \boldsymbol{x}_{23}^2}{\boldsymbol{x}_{12}^2 \boldsymbol{x}_{34}^2} \quad \text{and} \quad v = \frac{\boldsymbol{x}_{13}^2 \boldsymbol{x}_{24}^2}{\boldsymbol{x}_{12}^2 \boldsymbol{x}_{34}^2} \tag{5.3.72}$$

are the two independent dual $SL(2, C)$ invariants that can be formed from $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_4$. This implies that the integral $I_1 - I_2$ will be finite and that we can apply the dual conformal symmetry to conclude that

$$I_1 - I_2 = \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{23}^2 \boldsymbol{x}_{24}^2} I(u,\, v)\ , \tag{5.3.73}$$

for some function $I(u,\, v)$ of the conformal invariants. Next, we observe that both I_1 and I_2 are symmetric under the interchanges $2 \leftrightarrow 3$, producing

$$u \leftrightarrow \frac{u}{v}\ ,\ v \leftrightarrow \frac{1}{v} \text{ and } \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{23}^2 \boldsymbol{x}_{24}^2} \leftrightarrow v \frac{\boldsymbol{x}_{12}^2}{\boldsymbol{x}_{23}^2 \boldsymbol{x}_{24}^2}\ , \tag{5.3.74}$$

as well as $3 \leftrightarrow 4$ which gives

$$u \leftrightarrow v\ . \tag{5.3.75}$$

When applied to (5.3.73) these symmetries give

$$I(u,\, v) = vI\left(\frac{u}{v},\, \frac{1}{v}\right) \quad \text{and} \quad I(u,\, v) = I(v,\, u)\ , \tag{5.3.76}$$

which implies that

$$I(u,\, v) = (1+u+v)G(\xi_1,\, \xi_2)\ , \tag{5.3.77}$$

for some function G, where

$$\xi_1 = \frac{u+v+uv}{(1+u+v)^2} \quad \text{and} \quad \xi_2 = \frac{uv}{(1+u+v)^3} \tag{5.3.78}$$

are two independent invariants of (5.3.74) and (5.3.75).

The difference between the two integrals I_1 and I_2 is further restricted by noting that for some values of $\boldsymbol{x}_1,\, \ldots,\, \boldsymbol{x}_4$, their integrands coincide for all values of the integration variable. This occurs, for example, if $\boldsymbol{x}_1$ equals one of the other $\boldsymbol{x}_i$, translating to $G(1/4,\, 0) = 0$. In fact, it can be shown that the two integrands are equal for all $\boldsymbol{x}_I$ precisely when

$$\frac{u+v+uv}{(1+u+v)^2} = 1/4\ , \tag{5.3.79}$$

implying that $G(1/4,\, \xi_2) = 0$, for arbitrary ξ_2. As a consequence we can write

$$I(u,\, v) = (1+u+v)\left(4\frac{u+v+uv}{(1+u+v)^2} - 1\right) H(\xi_1,\, \xi_2) = \frac{2(u+v)-(u-v)^2-1}{1+u+v} H(\xi_1,\, \xi_2)\ , \tag{5.3.80}$$

where H must be finite, for all ξ_2, when $\xi_1 \to 1/4$.

It does not seem that we can obtain anything more from symmetry considerations alone. Still, from equations (5.3.73) and (5.3.80) we see that the form of I_1 is greatly restricted. It should then not come as a surprise that in fact $I_1 = I_2$, as can easily be checked numerically.

The dual symmetry can also be applied to restrict the form of more complicated integrals. For example, at the third iteration of the BFKL equation, integrals such as

$$I_3 = \frac{1}{\pi} \int \frac{d^2\boldsymbol{x}_I\, \boldsymbol{x}_{1I}^2}{\boldsymbol{x}_{2I}^2 \boldsymbol{x}_{3I}^2 \boldsymbol{x}_{4I}^2} \ln\left(\frac{\boldsymbol{x}_{4I}^2}{\boldsymbol{x}_{1I}^2}\right) \tag{5.3.81}$$

appear. If we define I_4 by exchanging $\frac{1}{\pi}\frac{\boldsymbol{x}_{1I}^2}{\boldsymbol{x}_{2I}^2 \boldsymbol{x}_{3I}^2 \boldsymbol{x}_{4I}^2}$ for the integrand of I_2, then $I_4 - I_3$ will once again be restricted by dual $SL(2, C)$ symmetry, symmetry under the exchange of $2 \leftrightarrow 3$, and the condition that the difference vanishes when (5.3.79) is satisfied.

Chapter 6

Conclusions

Even though it has been concerned with two very different areas of research - that of the spectrum of anomalous dimensions in $\mathcal{N}=4$ Super Yang Mills and that of scattering amplitudes in the Regge limit - this thesis has a common theme which is that of previously hidden symmetries, appearing in the planar limit. In the context of the β-deformed $\mathcal{N}=4$ theory, which inherits the planar integrability of the undeformed theory, we found a hidden symmetry of the spectrum of the deformed $SU(2)$ sector under the change $\beta \to \beta + 1/L$, if one relaxes the cyclicity constraint. As a check of this symmetry we used that the single magnon of momentum π could be obtained in the undeformed theory by analytically continuing the physical spectrum of twist-two operators. On the other hand, we also presented a new dual $SL(2, C)$ symmetry of the BFKL equation, governing the color-singlet exchange amplitude in the Regge limit, and showed that an apparent breaking of the symmetry by IR effects could be reabsorbed by a deformation of its representation, to leading order in the symmetry breaking. This symmetry was inspired by the dual conformal symmetry of $\mathcal{N}=4$ and can potentially explain the integrability of the generalized Leading Logarithmic Approximation of the Regge limit, in the planar limit.

Also, if we permit ourselves to be a bit imaginative, the two proposed symmetries may be more connected than first appears. Even though it is not at all clear how one would prove it, the dual $SL(2, C)$ symmetry could very well be a consequence of the dual conformal symmetry of $\mathcal{N}=4$ planar scattering amplitudes, which is an important component of their $PSU(2,2|4)$ Yangian symmetry. The dilatation operator has the same Yangian symmetry, so it is also related to the integrable model underlying the spectrum of anomalous dimensions. In the $SU(2)$ sector, the latter is also related to the spectrum of the β-deformed theory, since one can relax the cyclicity constraint in that case.

We have also discussed applications of the new symmetries. In the case of the β-deformed symmetry, we saw that it imposes constraints on how the first wrapping correction to the dilatation operator in the original $\mathcal{N} = 4$ theory should behave. These constraints enter when acting with the Hamiltonian defined by the dilatation operator on states which do not correspond to gauge invariant operators, but which are related by the symmetry to physical operators in the β-deformed theory. Since the action on non-physical states is also important if we want to understand the structure of the dilatation operator, I believe that a complete understanding of planar $\mathcal{N} = 4$ Super Yang Mills will be achieved at the same time as one understands the β-deformed theory. As for the dual $SL(2,\, C)$ symmetry of the BFKL equation, we saw that it can help us restrict the form of the two-dimensional integrals appearing in the high-energy limit.

As for problems that have been left open by the thesis, the obvious ones are to prove the symmetries that have been presented. For the $\beta \to \beta + 1/L$ symmetry this is not trivial, despite proposals, such as [171], for a Y-system providing the complete spectrum, since the symmetry only appears after relaxing the cyclicity constraint. And one must keep in mind that models incorporating the entire $PSU(2, 2|4)$ require the cyclicity constraint for consistency.

Proving the dual $SL(2,\, C)$ would seem to be even more difficult. One possibility is of course to try to derive it from the dual conformal symmetry of $\mathcal{N} = 4$ but the main obstacle to that approach is that the operations of projecting onto the singlet, and taking the planar limit do not commute. The object which appears more naturally in the color-ordered amplitudes exhibiting the dual conformal symmetry is the reggeized gluon. However, as we mentioned in [228], in the BFKL equation a different color projection simply results in a different numerical pre-factor multiplying the integral part of the Kernel, and will therefore also have a formal dual $SL(2, C)$. And indeed, Lipatov does find a dual $SL(2,\, C)$ symmetry in the octet channel in [233], although we are not sure how or if it is related to our symmetry. A worst case scenario would of course be that the formal dual $SL(2,\, C)$ symmetry for singlet exhange is what is left of Lipatov's exact octet channel symmetry, and that it is destroyed by IR effects. However, we did show that the commutation relations of the original and dual symmetries could be recovered to leading order in the symmetry breaking by deforming the dual algebra's representation. That this can be done is non-trivial and indicates that the symmetry may in fact hold to all orders.

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